

*Chapter 9

Stable Sets in Graphs

In this chapter we survey the results of the polyhedral approach to a particular \mathcal{NP} -hard combinatorial optimization problem, the stable set problem in graphs. (Alternative names for this problem used in the literature are vertex packing, or coclique, or independent set problem.) Our basic technique will be to look for various classes of inequalities valid for the stable set polytope, and then develop polynomial time algorithms to check if a given vector satisfies all these constraints. Such an algorithm solves a relaxation of the stable set problem in polynomial time, i. e., provides an upper bound for the maximum weight of a stable set. If certain graphs have the property that every facet of the stable set polytope occurs in the given family of valid inequalities, then, for these graphs, the stable set problem can be solved in polynomial time. It turns out that there are very interesting classes of graphs which are in fact characterized by such a condition, most notably the class of perfect graphs. Using this approach, we shall develop a polynomial time algorithm for the stable set problem for perfect graphs. So far no purely combinatorial algorithm has been found to solve this problem in polynomial time.

Let us mention that all algorithms presented in this chapter can be made strongly polynomial using Theorem (6.6.5), with the natural exception of the algorithm designed to prove Theorem (9.3.30), which optimizes a linear objective function over a nonpolyhedral set.

*9.1 Odd Circuit Constraints and t -Perfect Graphs

Throughout this chapter, $G = (V, E)$ denotes a graph with node set $V = \{1, 2, \dots, n\}$. Let $w : V \rightarrow \mathbb{Q}_+$ be any weighting of the nodes of G , and let $\alpha(G, w)$ denote the maximum weight of a stable set in G . It is well known that to determine $\alpha(G, w)$ is \mathcal{NP} -hard, even in the special case when $w = \mathbf{1}$ – see, for instance, GAREY and JOHNSON (1979).

Similarly as, say, in the case of matchings – see Sections 7.3 and 8.5 – we introduce the **stable set polytope**

$$\text{STAB}(G) := \text{conv}\{\chi^S \in \mathbb{R}^V \mid S \subseteq V \text{ stable set}\}$$

defined as the convex hull of the incidence vectors of all stable sets of nodes of G . Then $\alpha(G, w)$ is equal to the maximum value of the linear function $w^T x$ for $x \in \text{STAB}(G)$. For this observation to be of any use, however, we need

information about inequalities defining $\text{STAB}(G)$. So let us collect inequalities valid for $\text{STAB}(G)$, and see if they are enough to describe this polytope.

The following sets of linear inequalities are obviously all valid for $\text{STAB}(G)$:

$$(9.1.1) \quad x_i \geq 0 \quad \text{for all } i \in V,$$

$$(9.1.2) \quad x_i + x_j \leq 1 \quad \text{for all } ij \in E.$$

It is also easy to see that the integral solutions of (9.1.1), (9.1.2) are exactly the incidence vectors of stable sets of nodes of G . Theorem (8.2.8) implies :

(9.1.3) Proposition. *The inequalities (9.1.1), (9.1.2) are sufficient to describe $\text{STAB}(G)$ if and only if G is bipartite and has no isolated nodes.* □

We can take care of isolated nodes by adding, for each isolated node i , the inequality $x_i \leq 1$. So in particular, we see that the stable set problem for bipartite graphs can be solved using linear programming, since (9.1.1), (9.1.2) is an explicit system of linear inequalities, whose encoding length is polynomially bounded in the encoding length of G . (Combinatorial polynomial time methods for the stable set problem for bipartite graphs were mentioned in Section 8.2.)

The minimal graphs for which inequalities (9.1.1) and (9.1.2) are not sufficient to describe $\text{STAB}(G)$ are the odd circuits. In fact, if $G = (V, E)$ is an odd circuit, the point $(\frac{1}{2}, \dots, \frac{1}{2})^T \in \mathbb{R}^V$ satisfies all inequalities in (9.1.1) and (9.1.2) but is not in $\text{STAB}(G)$. This suggests a new class of inequalities valid for $\text{STAB}(G)$, the so-called **odd circuit constraints**:

$$(9.1.4) \quad \sum_{i \in V(C)} x_i \leq \frac{|V(C)| - 1}{2} \quad \text{for each odd circuit } C.$$

Let us call a graph **t -perfect** if (9.1.1), (9.1.2), and (9.1.4) are enough to describe $\text{STAB}(G)$ (the “ t ” stands for “trou”, the French word for hole). The study of these graphs was suggested by CHVÁTAL (1975). Although t -perfect graphs do not seem to occur in such an abundance as perfect graphs (to be described in the next section), there are some interesting classes of these graphs known.

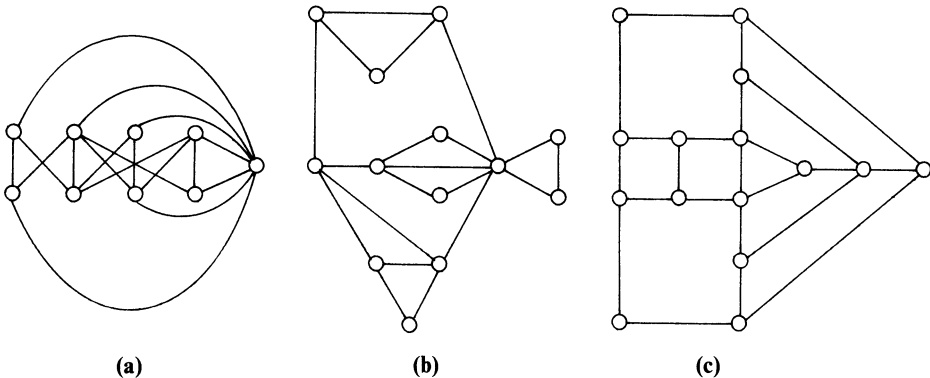


Figure 9.1

(9.1.5) Examples.

I. Bipartite graphs. This follows trivially from Proposition (9.1.3), as odd circuit inequalities do not occur at all.

II. Almost bipartite graphs. A graph is **almost bipartite** if it has a node v such that all odd circuits go through v (equivalently, $G - v$ is bipartite) – see Figure 9.1 (a) for an example. The t -perfectness of this class was shown by FONLUPT and UHRY (1982). It is trivial to check if a graph is almost bipartite, and also the stable set problem for such graphs is easily reduced to the stable set problem for bipartite graphs.

III. Series-parallel graphs. A graph is **series-parallel** if it can be obtained from a forest by repeated application of the following operations: adding an edge parallel to an existing edge and replacing an edge by a path. DIRAC (1952) and DUFFIN (1965) characterized these graphs as those containing no subdivision of K_4 . It is easy to check whether a graph is series-parallel. CHVÁTAL (1975) conjectured that series-parallel graphs are t -perfect. This was proved by BOULALA and UHRY (1979), who also designed a combinatorial polynomial time algorithm for the weighted stable set problem in series-parallel graphs. Series-parallel graphs include **cacti** which are graphs in which every block is either a bridge (i. e., an edge that is a cut) or a chordless circuit. A series-parallel graph is shown in Figure 9.1 (b).

IV. Nearly bipartite planar graphs. These are planar graphs in which at most two faces are bounded by an odd number of edges – see Figure 9.1 (c). Using a polynomial time planarity testing algorithm (HOPCROFT and TARJAN (1974)), these graphs can also be recognized in polynomial time. Their t -perfectness follows from the results of Gerards and Schrijver (see below).

V. Strongly t -perfect graphs. These are graphs that do not contain a subdivision of K_4 such that all four circuits corresponding to triangles in K_4 are odd. GERARDS and SCHRIJVER (1986) proved that these graphs are t -perfect. In fact, they proved that these graphs are characterized by the following property stronger than t -perfectness:

Let $a: V \rightarrow \mathbb{Z}$ and $b: V \rightarrow \mathbb{Z}$ be two integral weightings of the nodes, and let $c: E \rightarrow \mathbb{Z}$ and $d: E \rightarrow \mathbb{Z}$ be two integral weightings of the edges. Consider the inequalities

$$(9.1.6) \quad a_v \leq x_v \leq b_v \quad \text{for each } v \in V,$$

$$(9.1.7) \quad c_{uv} \leq x_u + x_v \leq d_{uv} \quad \text{for each } uv \in E,$$

and, for each circuit $C = (v_1, \dots, v_k)$ and each choice of $\varepsilon_i = \pm 1$ ($i = 1, \dots, k$), the inequality

$$(9.1.8) \quad \sum_{i=1}^k \frac{\varepsilon_i + \varepsilon_{i-1}}{2} x_{v_i} \leq \left\lfloor \frac{1}{2} \left(\sum_{\varepsilon_i=1} d_{v_i, v_{i+1}} - \sum_{\varepsilon_i=-1} c_{v_i, v_{i+1}} \right) \right\rfloor$$

taking indices mod k . Then the solution set of (9.1.6), (9.1.7), (9.1.8) has integral vertices.

So the inequalities (9.1.6), (9.1.7), (9.1.8) describe the convex hull of the integer solutions of (9.1.6), (9.1.7).

GERARDS, LOVÁSZ, SEYMOUR, SCHRIJVER and TRUEMPER (1987) proved that every strongly t -perfect graph can be “glued together” from almost bipartite graphs and nearly bipartite planar graphs. We do not go into the details of this decomposition, but remark that it yields combinatorial polynomial time procedures to recognize whether a graph is strongly t -perfect and to find a maximum weight stable set.

Not every t -perfect graph is strongly t -perfect, as is shown by the graph obtained from K_4 by subdividing the lines of a 4-circuit – see Figure 9.2. \square

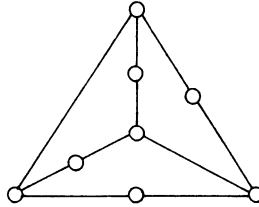


Figure 9.2

Let G be any graph. We set

$$(9.1.9) \quad \text{CSTAB}(G) := \{ x \in \mathbb{R}^V \mid x \text{ satisfies (9.1.1), (9.1.2), (9.1.4)} \}$$

(CSTAB stands for, say, **circuit-constrained stable set polytope**). The main result in this section is the following.

(9.1.10) Theorem. *The strong optimization problem for $\text{CSTAB}(G)$ can be solved in polynomial time for any graph G . Moreover, an optimum vertex solution can be found in polynomial time.*

By Theorem (6.4.9) and Lemma (6.5.1), it suffices to prove that the strong separation problem can be solved in polynomial time. Thus (9.1.10) is implied by the next lemma.

(9.1.11) Lemma. *There exists a polynomial time algorithm that, for any graph $G = (V, E)$ and for any vector $y \in \mathbb{Q}^V$, either*

- (a) *asserts that $y \in \text{CSTAB}(G)$, or*
- (b) *finds an inequality from (9.1.1), (9.1.2), or (9.1.4) violated by y .*

Proof. The inequalities in (9.1.1) and (9.1.2) are easily checked by substitution. So we may assume that $y \geq 0$ and that, for each edge $uv \in E$, $y_u + y_v \leq 1$.

Define, for each edge $e = uv \in E$, $z_e := 1 - y_u - y_v$. So $z_e \geq 0$. Then (9.1.4) is equivalent to the following set of inequalities :

$$(9.1.12) \quad \sum_{e \in C} z_e \geq 1 \quad \text{for each odd circuit } C.$$

If we view z_e as the “length” of edge e , then (9.1.12) says that the length of a shortest odd circuit is at least 1. But a shortest odd circuit can be found in polynomial time (see (8.3.6) and the remarks thereafter). This proves the lemma. \square

Since for t -perfect graphs $\text{CSTAB}(G) = \text{STAB}(G)$ holds, we can conclude:

(9.1.13) Corollary. *A maximum weight stable set in a t -perfect graph can be found in polynomial time.* \square

Theorem (9.1.10) implies that the property of t -perfectness is in $\text{co-}\mathcal{NP}$. In fact, to show that G is not t -perfect, it suffices to exhibit a vertex of $\text{CSTAB}(G)$ which is nonintegral, and since we can optimize over $\text{CSTAB}(G)$ in polynomial time, we can prove that the exhibited vector is indeed a vertex of $\text{CSTAB}(G)$ in polynomial time by Corollary (6.5.10). We do not know whether the problem of checking t -perfectness is in \mathcal{NP} or in \mathcal{P} .

*9.2 Clique Constraints and Perfect Graphs

Instead of the odd circuit constraints, it is also natural to consider the following system of so-called **clique constraints**:

$$(9.2.1) \quad x(Q) \leq 1 \quad \text{for all cliques } Q \subseteq V.$$

Note that (9.2.1) contains (9.1.2) as a special case, and also contains the triangle constraints from (9.1.4), but in general, (9.1.4) and (9.2.1) do not imply each other. The graphs G for which the constraints (9.1.1) and (9.2.1) suffice to describe $\text{STAB}(G)$ are called **perfect**.

While polyhedrally this is a natural way to arrive at the notion of perfect graphs, there are several equivalent definitions, some of them in terms of more elementary graph theory. Perfect graphs were introduced by BERGE (1961) as common generalizations of several nice classes of graphs (see below for these special cases).

To motivate these elementary definitions of perfect graphs, observe that each graph G satisfies the following inequalities for the clique number $\omega(G)$, the coloring number $\chi(G)$, the stability number $\alpha(G)$ and the clique covering number $\bar{\chi}(G)$:

$$(9.2.2) \quad \begin{aligned} \omega(G) &\leq \chi(G), \\ \alpha(G) &\leq \bar{\chi}(G). \end{aligned}$$

The pentagon C_5 shows that strict inequality can occur in both inequalities above. Now BERGE (1961,1962) called a graph G **perfect** if

$$(9.2.3) \quad \omega(G') = \chi(G')$$

holds for each induced subgraph G' of G .

We will give a list of known classes of perfect graphs later, but observe that, e. g., not only bipartite graphs (trivially), line graphs of bipartite graphs (by König's edge-coloring theorem (7.4.3)) and comparability graphs (trivially) are perfect, but so are their complements (by König's edge covering theorem (8.2.4), König's matching theorem (8.2.3) and Dilworth's theorem (8.3.19), respectively). These results made BERGE (1961,1962) conjecture that the complement of a perfect graph is perfect again. This was proved by LOVÁSZ (1972). Note that this result generalizes the theorems of König and Dilworth mentioned above.

To illustrate the ideas relating combinatorial and polyhedral properties of perfect graphs, we give the proof of the following theorem, which is a combination of results of FULKERSON (1970) and LOVÁSZ (1972).

Let $w: V \rightarrow \mathbb{Z}_+$ be any weighting of the nodes of the graph G , and let $\chi(G, w)$ denote the weighted chromatic number of G , i. e., the minimum number k of stable sets S_1, S_2, \dots, S_k such that each $i \in V$ is contained in w_i of these sets. Note that for $w = \mathbf{1}$ this number is just the chromatic number of G , for arbitrary w , it could be defined as the chromatic number of the graph obtained from G by replacing each $i \in V$ by a complete subgraph of w_i nodes.

(9.2.4) Theorem. *For any graph $G = (V, E)$, the following are equivalent.*

- (i) $\chi(G') = \omega(G')$ for each induced subgraph G' of G .
- (ii) $\chi(G, w) = \omega(G, w)$ for each weighting $w: V \rightarrow \mathbb{Z}_+$.
- (iii) $\text{STAB}(G)$ is determined by the constraints (9.1.1) and (9.2.1).
- (iv) The complement \bar{G} of G satisfies (i).
- (v) The complement \bar{G} of G satisfies (ii).
- (vi) The complement \bar{G} of G satisfies (iii).

Proof. (i) \Rightarrow (ii). We use induction on $w(V) = \sum_{v \in V} w(v)$. If $w \leq \mathbf{1}$ then (ii) specializes to just (i), so we may assume that there is a node $i_0 \in V$ such that $w(i_0) > 1$. Consider the weights

$$w'(i) := \begin{cases} w(i_0) - 1 & \text{if } i = i_0, \\ w(i) & \text{if } i \neq i_0. \end{cases}$$

Then by the induction hypothesis, $\chi(G, w') = \omega(G, w')$, that is, there exists a family S_1, S_2, \dots, S_N of stable sets such that each node i is contained in $w'(i)$ of them and $N = \omega(G, w')$. Since $w'(i_0) = w(i_0) - 1 \geq 1$, there is a stable set S_j containing the node i_0 , say $i_0 \in S_1$.

Now consider the weighting

$$w'' := w - \chi^{S_1}.$$

Let Q be a clique so that $w''(Q) = \omega(G, w'')$. If $S_1 \cap Q = \emptyset$ then

$$\begin{aligned} \omega(G, w'') &= w''(Q) = \sum_{i \in Q} w''(i) \leq \sum_{i \in Q} w'(i) = \sum_{j=1}^N |S_j \cap Q| \leq \sum_{j=2}^N 1 = N - 1 \\ &= \omega(G, w') - 1 \leq \omega(G, w) - 1. \end{aligned}$$

If $S_1 \cap Q \neq \emptyset$ then

$$\omega(G, w'') = w''(Q) \leq w(Q) - 1 \leq \omega(G, w) - 1.$$

So it follows that

$$\omega(G, w'') \leq \omega(G, w) - 1.$$

By the induction hypothesis, there is a family $S'_1, S'_2, \dots, S_{\omega(G, w'')}$ of stable sets in G such that each node i is contained in $w''(i)$ of them. Adding S_1 to this family, we see that $\chi(G, w) \leq \omega(G, w)$. Since the reverse inequality holds trivially, (ii) is proved.

(ii) \Rightarrow (iii). Let $x \in \mathbb{Q}^V$ be any vector satisfying (9.1.1) and (9.2.1). We show that $x \in \text{STAB}(G)$. Let q be the least common denominator of the entries in x . Then $qx \in \mathbb{Z}_+^V$ and (9.2.1) says that $\omega(G, qx) \leq q$. Hence by (ii), $\chi(G, qx) \leq q$, i. e., there exists a family S_1, S_2, \dots, S_q of stable sets such that each $i \in V$ is contained in exactly qx_i of them. In other words

$$qx = \sum_{j=1}^q \chi^{S_j}, \quad \text{i. e.,} \quad x = \frac{1}{q} \sum_{j=1}^q \chi^{S_j},$$

which shows that $x \in \text{STAB}(G)$.

(iii) \Rightarrow (iv). If $\text{STAB}(G)$ is determined by (9.1.1) and (9.2.1) then the same is true for every induced subgraph G' of G . So we only have to show that G can be partitioned into $\alpha(G)$ complete subgraphs. We use induction on $|V|$. Let F be the face of $\text{STAB}(G)$ spanned by all stable sets of size $\alpha(G)$; obviously, there is a facet of the form $x(Q) \leq 1$ containing F , where Q is a clique. But this means that for each maximum stable set S , $|S \cap Q| = \chi^S(Q) = 1$. Hence $\alpha(G - Q) < \alpha(G)$. By the induction hypothesis, $G - Q$ can be partitioned into $\alpha(G - Q)$ complete subgraphs. Adding Q to this system, we obtain a partition of G into $\alpha(G)$ complete subgraphs.

The implications (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (i) follow by interchanging the roles of G and \bar{G} . □

It follows from these characterizations that, for perfect graphs, the clique, stable set, coloring, and clique covering problems, as well as their weighted versions, belong to the class $\mathcal{NP} \cap \text{co-}\mathcal{NP}$.

In fact, as we shall see later, in the perfect graph case these problems belong to the class \mathcal{P} of polynomially solvable problems. So this implies the polynomial time solvability of these four problems for several classes of graphs which were shown to be perfect.

First we shall list a number of classes of perfect graphs discovered until now. Clearly, with each graph automatically the class of complementary graphs is associated, which are perfect again. At each of the classes we shall give references to polynomial algorithms developed for those graphs for the four combinatorial optimization problems mentioned above (in a combinatorial, non-ellipsoidal way).

One should keep in mind that for perfect graphs determining $\omega(G)$ is the same as determining $\chi(G)$. However, finding an explicit coloring of size $\chi(G)$ and finding an explicit maximum clique may be more complex (similarly, for $\alpha(G)$ and $\bar{\chi}(G)$). Moreover, the clique problem (coloring problem) for a class of graphs is equivalent to the stable set problem (clique covering problem) for the class of complementary graphs.

Beside the four combinatorial optimization problems mentioned, there is the **recognition problem** for a class \mathcal{C} of graphs: given a graph G , does G belong to \mathcal{C} ? Clearly, one can speak of this problem being in \mathcal{NP} , in $\text{co-}\mathcal{NP}$, in \mathcal{P} , or \mathcal{NP} -complete. With each of the classes below we shall also discuss what is known on the complexity of the corresponding recognition problem. Obviously, the recognition problem for a class of graphs is equivalent to the recognition problem for the class of complements.

The status of the recognition problem of the class of all perfect graphs is unknown. It is well-known that this problem belongs to $\text{co-}\mathcal{NP}$. That is, one can prove in polynomial time that a given graph is not perfect. This would also follow directly from the strong perfect graph conjecture, posed by BERGE (1962), which is still unsolved.

(9.2.5) Strong Perfect Graph Conjecture. *A graph is perfect if and only if it does not contain an odd circuit of length at least five, or its complement, as an induced subgraph.* □

The content of this conjecture is that the minimal (under taking induced subgraphs) imperfect graphs (these graphs are also called critically imperfect graphs) are the odd circuits of length at least five and their complements.

We now give a list of classes of perfect graphs. Note that each class is closed under taking induced subgraphs. In these notes, by perfectness we mean that (9.2.3) is satisfied. Since a number of perfectness results appeared before Theorem (9.2.4) was established, we mention the results for classes of graphs and their complements separately.

(9.2.6) Classes of perfect graphs.

I. Bipartite graphs, and their complements. Bipartite graphs are trivially perfect. The perfectness of their complements follows from König's edge covering theorem (8.2.4). For bipartite graphs, the weighted clique and coloring problems are easily polynomially solvable, while the weighted stable set and clique covering problems were shown to be polynomially solvable by KUHN (1955) and FORD and FULKERSON (1956) – cf. Section 8.2. The recognition problem for bipartite graphs is easily solvable in polynomial time.

II. Line graphs of bipartite graphs, and their complements. The perfectness of line graphs of bipartite graphs follows from König's edge-coloring theorem (7.4.3), and that of their complements from König's Matching Theorem (8.2.3). For line graphs of bipartite graphs, the weighted clique problem is trivial. A polynomial algorithm for the unweighted coloring problem follows from König's proof in KÖNIG (1916). The weighted case follows from the proof of Satz 15 in Kapitel

XI given in KÖNIG (1936). Polynomial algorithms for the weighted stable set and clique covering problems were given by KUHN (1955) and FORD and FULKERSON (1956) – cf. Section 8.2. VAN ROOIJ and WILF (1965) showed that line graphs can be recognized and that their “ancestors” can be reconstructed in polynomial time. Hence, the recognition problem for line graphs of bipartite graphs is polynomially solvable.

III. Interval graphs, and their complements. A graph is an **interval graph** if its nodes can be represented by intervals on the real line such that two nodes are adjacent if and only if the corresponding intervals have a nonempty intersection. The perfectness of interval graphs follows from Dilworth’s theorem (8.3.19). The perfectness of their complements was observed by Gallai. The polynomial solvability of the weighted clique, stable set, coloring, and clique covering problems is not difficult. The recognition problem can be solved in polynomial time (FULKERSON and GROSS (1965)), even in linear time – see BOOTH and LUEKER (1976) (see LEKKERKERKER and BOLAND (1962) and GILMORE and HOFFMAN (1964) for good characterizations). A graph is an interval graph if and only if it is triangulated and its complement is a comparability graph – see IV and V below.

IV. Comparability graphs, and their complements. A graph is a **comparability graph** if its edges can be oriented to obtain a transitive acyclic digraph $D = (V, A)$, i. e., a digraph satisfying: *if $(u, v) \in A$ and $(v, w) \in A$ then $(u, w) \in A$* . The perfectness of comparability graphs is easy, while that of their complements is equivalent to Dilworth’s theorem (8.3.19). For comparability graphs polynomial algorithms for the weighted clique and coloring problems are trivial. Polynomial algorithms for the weighted stable set and clique covering problems can be easily derived from min-cost flow algorithms. Such an algorithm can also be derived from any maximum flow algorithm by the construction of FORD and FULKERSON (1962). The recognition problem for comparability graphs is polynomially solvable by the method of GALLAI (1967) (membership in \mathcal{NP} is trivial, while membership in co- \mathcal{NP} was shown by GHOUILA-HOURI (1962) and GILMORE and HOFFMAN (1964)).

Note that comparability graphs include bipartite graphs and complements of interval graphs. Another subclass is formed by the **permutation graphs**, which can be defined as those comparability graphs whose complement is again a comparability graph – see EVEN, PNUELI and LEMPEL (1972).

V. Triangulated graphs, and their complements. A graph is a **triangulated** (or chordal) **graph** if it does not contain a circuit of length at least four as induced subgraph. BERGE (1960) proved perfectness of triangulated graphs, and HAJNAL and SURÁNYI (1958) proved perfectness of their complements. DIRAC (1961) showed that a triangulated graph always contains a node all of whose neighbors form a clique. This yields that an undirected graph is triangulated if and only if its edges can be oriented to obtain an acyclic directed graph $D = (V, A)$ satisfying: *if $(u, v) \in A$ and $(u, w) \in A$ then $(v, w) \in A$ or $(w, v) \in A$* . Dirac’s theorem also gives that a graph is triangulated if and only if it is the intersection graph of a collection of subtrees of a tree. The polynomial solvability of the weighted clique, stable set, coloring, and clique covering problems was shown by GAVRIL (1972) and FRANK (1976) (here again Dirac’s result is used). Also the recognition

problem can be shown to be polynomially solvable with Dirac's result. For linear time algorithms – see LUEKER (1974), ROSE and TARJAN (1975), and ROSE, TARJAN and LUEKER (1976). Triangulated graphs include interval graphs. They also include **split graphs**, which are graphs whose node set is the union of a clique and a stable set. These graphs can be characterized by the property that they are triangulated and their complements are triangulated.

VI. Parity graphs, and their complements. A graph is a **parity graph** if each odd circuit of length at least five has two crossing chords. The perfectness of this class of graphs was shown by E. Olaru – see SACHS (1970). The polynomial solvability of the weighted clique, stable set, coloring, and clique covering problems was shown by BURLET and UHRY (1982), who also proved the polynomial time solvability of the recognition problem for parity graphs. Clearly, parity graphs include bipartite graphs. They also include **line perfect** graphs (which are graphs G for which the line graph $L(G)$ is perfect), since TROTTER (1977) showed that a graph is line perfect if and only if it does not contain an odd circuit of length larger than three.

VII. Gallai graphs, and their complements. A graph is a **Gallai graph** (or *i*-triangulated graph) if each odd circuit of length at least five has two noncrossing chords. The perfectness of this class of graphs was shown by GALLAI (1962). BURLET and FONLUPT (1984) gave combinatorial algorithms that solve the unweighted versions of the four basic problems in polynomial time, and they also showed that Gallai graphs can be recognized in polynomial time. The recognition problem was also solved by WHITESIDES (1984). Gallai graphs include bipartite graphs, interval graphs, line perfect graphs, and triangulated graphs.

VIII. Meyniel graphs, and their complements. A graph is a **Meyniel graph** if each odd circuit of length at least five has two chords. The perfectness of this class of graphs was shown by MEYNIEL (1976). Again BURLET and FONLUPT (1984) found polynomial time combinatorial algorithms for the unweighted versions of all problems in question. Meyniel graphs include bipartite graphs, interval graphs, triangulated graphs, parity graphs, and Gallai graphs.

IX. Perfectly orderable graphs, and their complements. A graph is a **perfectly orderable** graph if its edges can be oriented to obtain an acyclic directed graph (V, A) with no induced subgraph isomorphic to $(\{t, u, v, w\}, \{(t, u), (u, v), (w, v)\})$. The perfectness of perfectly orderable graphs was shown by CHVÁTAL (1984). The recognition problem for perfectly orderable graphs trivially is in \mathcal{NP} , but no polynomial time algorithm for recognizing these graphs is known. Once an orientation with the above property is found, a maximum clique and a minimum coloring can be obtained by a greedy algorithm – see CHVÁTAL (1984). Combinatorial polynomial time algorithms for the problems in question are not known. Perfectly orderable graphs include bipartite graphs, interval graphs, complements of interval graphs, comparability graphs, triangulated graphs, and complements of triangulated graphs. CHVÁTAL, HOANG, MAHADEV and DE WERRA (1985) showed that a number of further classes of graphs are perfectly orderable.

X. Unimodular graphs, and their complements. An undirected graph is **unimodular** if the matrix whose rows are the incidence vectors of its maximal cliques is

totally unimodular. The perfectness of unimodular graphs follows from results of BERGE and LAS VERGNAS (1970). The perfectness of complements of unimodular graphs follows from the results of HOFFMAN and KRUSKAL (1956) on totally unimodular matrices. W. H. Cunningham (private communication) observed that the recognition problem for unimodular graphs can be solved in polynomial time using the algorithm of SEYMOUR (1980a) to recognize totally unimodular matrices. The stable set and clique cover problems for unimodular graphs can be written as explicit linear programs with totally unimodular matrix and therefore they are solvable by any linear programming algorithm. MAURRAS, TRUEMPER and AKGÜL (1981) designed a special algorithm for such linear programs while Bland and Edmonds (unpublished) remarked that Seymour's decomposition of totally unimodular matrices yields a combinatorial algorithm for such LP's. The clique problem is trivial, since, by a result of HELLER (1957), there are at most $\binom{n}{2}$ maximal cliques. Also the coloring problem can be reduced to linear programs over totally unimodular matrices. Unimodular graphs include bipartite graphs and their line graphs, interval graphs, and the class of graphs which do not contain an odd circuit of length at least five, the complement of such a circuit, or a $K_4 - e$ as an induced subgraph.

XI. Parthasarathy-Ravindra graphs, and their complements. A graph is a **Parthasarathy-Ravindra graph** if it does not contain a claw, an odd circuit of length at least five, or the complement of such an odd circuit as an induced subgraph. Perfectness of these graphs was proved by PARTHASARATHY and RAVINDRA (1976). SBIHI (1978, 1980) and MINTY (1980) showed that the stable set problem can be solved in polynomial time for claw-free (not necessarily perfect) graphs (this class of graphs contains the line graphs, and thus the matching problem is included). Minty's algorithm extends to the weighted case. For Parthasarathy-Ravindra graphs polynomial time algorithms for the cardinality coloring problem resp. the weighted clique and weighted clique covering problems were given by HSU (1981) resp. HSU and NEMHAUSER (1981, 1982). Parthasarathy-Ravindra graphs can be recognized in polynomial time by a method of V. Chvátal and N. Sbihi (personal communication). This class of graphs includes line graphs of bipartite graphs and complements of bipartite graphs.

XII. Strongly perfect graphs, and their complements. A graph is **strongly perfect** if every induced subgraph contains a stable set of nodes that meets all its (inclusionwise) maximal cliques. Perfectness of these graphs was observed by BERGE and DUCHET (1984) who also proved that the recognition problem for strongly perfect graphs is in $\text{co-}\mathcal{NP}$. A polynomial time algorithm for recognizing these graphs is not known. Moreover, combinatorial polynomial time algorithms for the four optimization problems in question have not been found yet. The class of strongly perfect graphs contains Meyniel graphs, comparability graphs, and perfectly orderable graphs.

XIII. Weakly triangulated graphs. A graph is **weakly triangulated** if it does neither contain a circuit of length at least five nor the complement of a circuit of length at least five as an induced subgraph. HAYWARD (1985) proved that weakly triangulated graphs are perfect. The recognition problem for weakly triangulated graphs is trivially in $\text{co-}\mathcal{NP}$. No polynomial time algorithm for the recognition

problem, and no combinatorial polynomial time algorithm for any of the four optimization problems for weakly triangulated graphs is known. Triangulated graphs and their complements are weakly triangulated.

XIV. Quasi parity graphs, and their complements. A graph is a **quasi parity** graph if every induced subgraph G' that is not a clique has two nodes that are not connected by a chordless path of odd length in G' . Perfectness of quasi parity graphs was shown by MEYNIEL (1985). Berge (personal communication) observed that this class of graphs can be enlarged by the graphs for which every induced subgraph G' with at least two nodes has the property that either G' or the complement of G' has two nodes that are not connected by a chordless path of odd length. The recognition problem for this class (resp. these two classes) of graphs is in $\text{co-}\mathcal{NP}$ but is not known to be in \mathcal{P} . No combinatorial polynomial time algorithm for the four optimization problems is known. Quasi parity graphs include Meyniel graphs, and hence parity graphs and Gallai graphs. \square

More information about perfect graphs and their properties can be found in GOLUMBIC (1980), LOVÁSZ (1983a), and the collection of papers BERGE and CHVÁTAL (1984).

Now we continue our study of the inequality systems (9.1.1) and (9.2.1). Let $G = (V, E)$ be any graph. We set

$$(9.2.7) \quad \text{QSTAB}(G) := \{x \in \mathbb{R}^V \mid x \text{ satisfies (9.1.1) and (9.2.1)}\},$$

and call it the **clique-constrained stable set polytope**. (In the literature $\text{QSTAB}(G)$ is often called the “fractional stable set polytope”.) The following immediate observation relates stable set properties of a graph G and its complement \bar{G} :

(9.2.8) Proposition. *The antiblocker of $\text{STAB}(G)$ is $\text{QSTAB}(\bar{G})$, and the antiblocker of $\text{QSTAB}(G)$ is $\text{STAB}(\bar{G})$.* \square

By analogy with the preceding section, one may expect that the optimization problem over $\text{QSTAB}(G)$ is polynomially solvable. This expectation is even more justified since $\text{QSTAB}(G)$ has nice and “easily recognizable” facets. However GRÖTSCHEL, LOVÁSZ and SCHRIJVER (1981) proved the following :

(9.2.9) Theorem. *The optimization problem for $\text{QSTAB}(G)$ is \mathcal{NP} -hard.*

Although we usually do not prove \mathcal{NP} -hardness results in this book, we sketch the proof of Theorem (9.2.9) because it makes use of the ellipsoid method.

Proof. (Sketch). The optimization problem for $\text{QSTAB}(G)$ is polynomially equivalent to the optimization problem for its antiblocker (see Exercise (6.5.18)), which is just the polytope $\text{STAB}(\bar{G})$. So this problem is equivalent to the stable set problem for \bar{G} , which is \mathcal{NP} -hard for general graphs. \square

In the next section, however, we introduce an infinite class of valid inequalities for $\text{STAB}(G)$ which includes the clique constraints (9.2.1) and for which the separation problem can be solved in polynomial time.

Antiblockers of Hypergraphs

Before doing this let us come back – just for a side remark – to an issue raised in Section 8.1. We have seen there how the notion of blocking hypergraphs and its relation to blocking polyhedra provides a common framework for quite a number of combinatorial optimization problems. We now introduce the analogous notion of antiblocking hypergraphs and relate it to antiblocking polyhedra. It will turn out that this is equivalent to the study of perfect graphs. These results are due to FULKERSON (1971, 1972).

Given a hypergraph H , its **antiblocker** $ABL(H)$ is the set of all inclusionwise maximal subsets of $\cup H$ that intersect each edge of H in at most one element. So $ABL(H)$ is a clutter. Define a graph $G(H)$ on the node set $\cup H$ by connecting two nodes if and only if they are contained in an edge of H . Then $ABL(H)$ is the collection of maximal stable sets of $G(H)$. In contrast to (8.1.1), which states that $BL(BL(H)) = H$ holds for each clutter H , there are clutters H for which $ABL(ABL(H)) \neq H$; e. g., take $H = \{ \{1, 2\}, \{2, 3\}, \{1, 3\} \}$. In fact, $ABL(ABL(H)) = H$ if and only if H is the clutter of all maximal stable sets of a graph. Such an H is called a **conformal clutter**. In this case, $ABL(H)$ is just the clutter of maximal cliques of the graph, i. e., the clutter of maximal stable sets of the complementary graph. Note that this complementary graph is just $G(H)$.

In analogy to blocking theory we consider the antidominant in $\mathbb{R}_+^{\cup H}$ – see Section 0.1 – of incidence vectors of a clutter H and denote it by $\text{admt}(H)$, i. e.,

$$\text{admt}(H) := (\text{conv}\{ \chi^E \in \mathbb{R}^{\cup H} \mid E \in H \} - \mathbb{R}_+^{\cup H}) \cap \mathbb{R}_+^{\cup H} .$$

If H is conformal then this is just the stable set polytope of the complement $\overline{G(H)}$ of $G(H)$. Again valid inequalities for $\text{admt}(H)$ are

$$(9.2.10) \quad \begin{array}{ll} \text{(a)} & x_v \geq 0 \quad \text{for all } v \in \cup H , \\ \text{(b)} & x(F) \leq 1 \quad \text{for all } F \in ABL(H) . \end{array}$$

The question when these inequalities suffice to describe $\text{admt}(H)$ is completely answered by the following theorem which can easily be derived from the previous results on perfect graphs.

(9.2.11) Theorem. *For a clutter H the following are equivalent:*

- (i) (9.2.10) suffices to describe $\text{admt}(H)$, i. e., (9.2.10) is TPI.
- (ii) (9.2.10) is TDI.
- (iii) H is conformal and $G(H)$ is perfect.
- (iv) The following system is TPI: $x_v \geq 0$ for all $v \in \cup H$, $x(F) \leq 1$ for all $F \in H$.
- (v) The system in (iv) is TDI.

□

So the analogue of Lehman’s theorem (8.1.5) holds even with dual integrality. The above shows that instead of antiblocking hypergraphs it suffices to study stable sets in graphs.

*9.3 Orthonormal Representations

The approach described in this section is based on LOVÁSZ (1979). The extension to the weighted case and a discussion of its algorithmic implications can be found in GRÖTSCHEL, LOVÁSZ and SCHRIJVER (1981, 1984b, 1986).

Let $G = (V, E)$ be a graph. An **orthonormal representation of G** is a sequence $(u_i \mid i \in V)$ of $|V|$ vectors $u_i \in \mathbb{R}^N$, where N is some positive integer, such that $\|u_i\| = 1$ for all $i \in V$ and $u_i^T u_j = 0$ for all pairs i, j of nonadjacent vertices. Trivially, every graph has an orthonormal representation (just take all the vectors u_i mutually orthogonal in \mathbb{R}^V). Figure 9.3 shows a less trivial orthonormal representation of the pentagon C_5 in \mathbb{R}^3 . It is constructed as follows. Consider an umbrella with five ribs of unit length (representing the nodes of C_5) and open it in such a way that nonadjacent ribs are orthogonal. Clearly, this can be achieved in \mathbb{R}^3 and gives an orthonormal representation of the pentagon. The central handle (of unit length) is also shown.

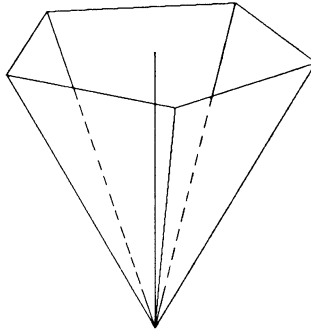


Figure 9.3

Let $(u_i \mid i \in V)$, $u_i \in \mathbb{R}^N$, be any orthonormal representation of G and let $c \in \mathbb{R}^N$ with $\|c\| = 1$. Then for any stable set $S \subseteq V$, the vectors u_i , $i \in S$, are mutually orthogonal and hence,

$$(9.3.1) \quad \sum_{i \in S} (c^T u_i)^2 \leq 1 .$$

Since $\sum_{i \in V} (c^T u_i)^2 x_i^S = \sum_{i \in S} (c^T u_i)^2$, we see that the inequality

$$(9.3.2) \quad \sum_{i \in V} (c^T u_i)^2 x_i \leq 1$$

holds for the incidence vector $\chi^S \in \mathbb{R}^V$ of any stable S set of nodes of G . Thus, (9.3.2) is a valid inequality for $\text{STAB}(G)$ for any orthonormal representation $(u_i \mid i \in V)$ of G , where $u_i \in \mathbb{R}^N$, and any unit vector $c \in \mathbb{R}^N$. We shall call (9.3.2) the **orthonormal representation constraints** for $\text{STAB}(G)$.

If Q is any clique of G , we can define an orthonormal representation $(u_i \mid i \in V)$ as follows. Let $\{u_i \mid i \in V \setminus Q\} \cup \{c\}$ be mutually orthogonal unit vectors and set $u_j = c$ for $j \in Q$. Then the constraint (9.3.2) is just the clique constraint (9.2.1)

determined by Q . The orthonormal representation of C_5 depicted in Figure 9.3, with c pointing along its axis of symmetry, yields the inequality $\sum_{i \in C_5} \frac{1}{\sqrt{5}} x_i \leq 1$ or $\sum_{i \in C_5} x_i \leq \sqrt{5}$, which is not as strong as the odd circuit constraint but is not implied by the clique constraints. For any graph $G = (V, E)$, set

$$(9.3.3) \quad \text{TH}(G) := \{x \in \mathbb{R}^V \mid x \geq 0 \text{ and } x \text{ satisfies all orthonormal representation constraints (9.3.2)}\}.$$

(TH comes from the function “ \mathcal{G} ” to be studied below.) $\text{TH}(G)$ is the intersection of infinitely many halfspaces, so $\text{TH}(G)$ is a convex set. From the remarks above, it follows that

$$(9.3.4) \quad \text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G).$$

In general, $\text{TH}(G)$ is not a polytope. In fact, we will prove later that $\text{TH}(G)$ is a polytope if and only if G is perfect. The most important property of $\text{TH}(G)$ – for our purposes – is that one can optimize any linear objective function over $\text{TH}(G)$ in polynomial time. The algorithm to achieve this depends on a number of somewhat involved formulas for the maximum value of such an objective function.

Given a graph $G = (V, E)$ and weights $w \in \mathbb{R}_+^V$, we are going to study the value

$$(9.3.5) \quad \mathcal{G}(G, w) := \max\{w^T x \mid x \in \text{TH}(G)\}.$$

Let us (temporarily) introduce a number of further values (which eventually will all turn out to be equal to $\mathcal{G}(G, w)$).

Where $(u_i \mid i \in V)$, $u_i \in \mathbb{R}^N$, ranges over all orthonormal representations of G and $c \in \mathbb{R}^N$ over all vectors of unit length, let

$$(9.3.6) \quad \mathcal{G}_1(G, w) := \min_{\{c, (u_i)\}} \max_{i \in V} \frac{w_i}{(c^T u_i)^2}.$$

The quotient in (9.3.6) has to be interpreted as follows. If $w_i = 0$ then we take $w_i / (c^T u_i)^2 = 0$, even if $c^T u_i = 0$. If $w_i > 0$ but $c^T u_i = 0$ then we take $w_i / (c^T u_i)^2 = +\infty$. For notational convenience we introduce a number of further sets associated with the graph $G = (V, E)$:

$$(9.3.7) \quad \begin{aligned} \mathcal{S} &:= \{A \in \mathbb{R}^{V \times V} \mid A \text{ symmetric}\}, \\ \mathcal{M} &:= \{B = (b_{ij}) \in \mathcal{S} \mid b_{ij} = 0 \text{ for all } i, j \text{ adjacent in } G\}, \\ \mathcal{M}^\perp &:= \{A = (a_{ij}) \in \mathcal{S} \mid a_{ii} = 0 \text{ for all } i \in V \text{ and} \\ &\quad a_{ij} = 0 \text{ for all } i, j \text{ nonadjacent in } G\}, \\ \mathcal{D} &:= \{A \in \mathcal{S} \mid A \text{ positive semidefinite}\}. \end{aligned}$$

Clearly, \mathcal{M} is a linear subspace of the space \mathcal{S} of symmetric $|V| \times |V|$ -matrices, and \mathcal{M}^\perp is the orthogonal complement of \mathcal{M} in \mathcal{S} . For $w \in \mathbb{R}_+^V$, let moreover

$$(9.3.8) \quad \begin{aligned} \bar{w} &:= (\sqrt{w_i} \mid i \in V) \in \mathbb{R}^V, \\ W &:= \bar{w} \bar{w}^T = (\sqrt{w_i w_j} \mid i, j \in V) \in \mathcal{S}. \end{aligned}$$

Define

$$(9.3.9) \quad \mathfrak{g}_2(G, w) := \min\{\Lambda(A + W) \mid A \in \mathcal{M}^\perp\},$$

where $\Lambda(D)$ denotes the largest eigenvalue of D , and

$$(9.3.10) \quad \mathfrak{g}_3(G, w) := \max\{\overline{w}^T B \overline{w} \mid B \in \mathcal{D} \cap \mathcal{M} \text{ and } \text{tr}(B) = 1\}.$$

Finally, let $(v_i \mid i \in V)$ with $v_i \in \mathbb{R}^N$ range over all orthonormal representations of the complementary graph \overline{G} , and $d \in \mathbb{R}^N$ over all vectors of Euclidean norm 1; define

$$(9.3.11) \quad \mathfrak{g}_4(G, w) := \max_{\{d, (v_i)\}} \sum_{i \in V} (d^T v_i)^2 w_i.$$

We now show the following result.

(9.3.12) Theorem. *For every graph $G = (V, E)$ and every $w \in \mathbb{R}_+^V$,*

$$\mathfrak{g}(G, w) = \mathfrak{g}_1(G, w) = \mathfrak{g}_2(G, w) = \mathfrak{g}_3(G, w) = \mathfrak{g}_4(G, w).$$

Proof. First we remark that the theorem trivially holds if $w = 0$. So we may assume $w \neq 0$. The proof will consist of showing $\mathfrak{g} \leq \mathfrak{g}_1 \leq \mathfrak{g}_2 \leq \mathfrak{g}_3 \leq \mathfrak{g}_4 \leq \mathfrak{g}$. First we show

$$(9.3.13) \quad \mathfrak{g}(G, w) \leq \mathfrak{g}_1(G, w).$$

Choose a vector $x \in \text{TH}(G)$ that maximizes the linear objective function $w^T x$. Let $(u_i \mid i \in V)$ with $u_i \in \mathbb{R}^N$ be any orthonormal representation of G and $c \in \mathbb{R}^N$ any vector with $\|c\| = 1$. Then by (9.3.5) and (9.3.2)

$$\begin{aligned} \mathfrak{g}(G, w) &= w^T x = \sum_{i \in V} w_i x_i \\ &\leq \left(\max_i \frac{w_i}{(c^T u_i)^2} \right) \sum_{i \in V} (c^T u_i)^2 x_i \\ &\leq \max_i \frac{w_i}{(c^T u_i)^2}. \end{aligned}$$

Here we have used the convention that if $c^T u_i = 0$, then $w_i / (c^T u_i)^2 = +\infty$ if $w_i > 0$, while $w_i / (c^T u_i)^2 = 0$ if $w_i = 0$. By (9.3.6), this proves (9.3.13).

Now we show

$$(9.3.14) \quad \mathfrak{g}_1(G, w) \leq \mathfrak{g}_2(G, w).$$

Choose a matrix $A \in \mathcal{M}^\perp$, and set $t := \Lambda(A + W)$. (Note that $t > 0$ since $\text{tr}(A + W) = \text{tr}(W) > 0$.) Then the matrix $tI - A - W$ is positive semidefinite. And so by (0.1.4) we can write $tI - A - W = X^T X$ with some matrix $X \in \mathbb{R}^{V \times V}$. Let

$x_i \in \mathbb{R}^V$ denote the column of X corresponding to $i \in V$. Then our definitions imply

$$(9.3.15) \quad x_i^T x_i = t - w_i \quad \text{for all } i \in V$$

and

$$(9.3.16) \quad x_i^T x_j = -\sqrt{w_i w_j} \quad \text{for all } i, j \text{ nonadjacent in } G.$$

Let $c \in \mathbb{R}^V$ be a vector with $\|c\| = 1$ and orthogonal to all x_i , $i \in V$ (such a vector exists since X is singular), and consider the vectors $u_i := \sqrt{w_i/t}c + t^{-1/2}x_i$. Then for each $i \in V$, we obtain from (9.3.15)

$$u_i^T u_i = \frac{w_i}{t}c^T c + \frac{1}{t}x_i^T x_i = 1$$

and from (9.3.16) for any two nonadjacent nodes i, j ,

$$u_i^T u_j = \frac{\sqrt{w_i w_j}}{t}c^T c + \frac{1}{t}x_i^T x_j = 0.$$

Hence $(u_i \mid i \in V)$ is an orthonormal representation of G , and so by (9.3.6) and the definition of u_i

$$\vartheta_1(G, w) \leq \max_{i \in V} \frac{w_i}{(c^T u_i)^2} = \max_{i \in V} \frac{w_i}{w_i/t} = t = \Lambda(A + W).$$

Since this holds for each $A \in \mathcal{M}^\perp$, by (9.3.9) this proves (9.3.14).

Next we show

$$(9.3.17) \quad \vartheta_2(G, w) \leq \vartheta_3(G, w).$$

This relation is the heart of the proof; it provides the good characterization of $\vartheta(G, w)$ by going from “min” to “max”. By the definition of $\vartheta_3 := \vartheta_3(G, w)$ in (9.3.10) the inequality

$$\bar{w}^T B \bar{w} \leq \vartheta_3 \cdot \text{tr}(B)$$

is valid for all matrices $B \in \mathcal{D} \cap \mathcal{M}$. But this can be viewed as a linear inequality for B in the space \mathcal{S} , namely the inequality can be written as

$$(W - \vartheta_3 I) \bullet B \leq 0,$$

where “ \bullet ” denotes the Euclidean inner product in $\mathbb{R}^{V \times V}$, i. e., $A \bullet B = \text{tr}(A^T B)$. Note that \mathcal{D} and \mathcal{M} are cones. So $\mathcal{D} \cap \mathcal{M}$ is a cone, and recall that the cone polar to this cone in the linear space \mathcal{S} is $(\mathcal{D} \cap \mathcal{M})^\circ := \{A \in \mathcal{S} \mid A \bullet B \leq 0 \text{ for all } B \in \mathcal{D} \cap \mathcal{M}\}$. Moreover, we know that $(\mathcal{D} \cap \mathcal{M})^\circ = \mathcal{D}^\circ + \mathcal{M}^\circ$; hence the inequality above gives

$$W - \vartheta_3 I \in \mathcal{D}^\circ + \mathcal{M}^\circ.$$

It is easy to see that $\mathcal{D}^\circ = -\mathcal{D}$ and $\mathcal{M}^\circ = \mathcal{M}^\perp$, and so we can write

$$W - \mathfrak{g}_3 I = -D - A,$$

where $D \in \mathcal{D}$ and $-A \in \mathcal{M}^\perp$. So $\mathfrak{g}_3 I - (A + W)$ is positive semidefinite, and hence \mathfrak{g}_3 is not smaller than the largest eigenvalue of $A + W$. Thus by definition (9.3.9), we have

$$\mathfrak{g}_3 \geq \Lambda(A + W) \geq \mathfrak{g}_2(G, w),$$

which proves (9.3.17).

We now show

$$(9.3.18) \quad \mathfrak{g}_3(G, w) \leq \mathfrak{g}_4(G, w).$$

Choose $B \in \mathcal{D} \cap \mathcal{M}$ such that $\text{tr}(B) = 1$ and $\bar{w}^T B \bar{w} = \mathfrak{g}_3(G, w) =: \mathfrak{g}_3$. Since B is positive semidefinite, we can write it in the form $B = Y^T Y$ with $Y \in \mathbb{R}^{V \times V}$ – see (0.1.4). Let $e_i \in \mathbb{R}^V$ be the incidence vector of the singleton $\{i\}$, $i \in V$. Then $y_i := Y e_i$ is the i -th column of Y . Let $P := \{i \in V \mid y_i \neq 0\}$ and set $v_i := \frac{1}{\|y_i\|} y_i$ for all $i \in P$. Moreover, choose an orthonormal basis of $(\text{lin}\{v_i \mid i \in P\})^\perp$ and let for each $i \in V \setminus P$ one of the elements of this basis represent i ; call this vector v_i . Since $y_i^T y_j = b_{ij} = 0$ for all $i, j \in V$ that are adjacent, we see that $(v_i \mid i \in V)$ is an orthonormal representation of \bar{G} . Furthermore, $(Y \bar{w})^T Y \bar{w} = \bar{w}^T Y^T Y \bar{w} = \bar{w}^T B \bar{w} = \mathfrak{g}_3$ and hence $d := \frac{1}{\sqrt{\mathfrak{g}_3}} Y \bar{w}$ is a vector of unit length. Moreover, for $i \in P$, we have

$$d^T v_i = \frac{1}{\sqrt{\mathfrak{g}_3} \|y_i\|} \bar{w}^T Y^T Y e_i = \frac{1}{\sqrt{\mathfrak{g}_3} \|y_i\|} \bar{w}^T B e_i,$$

and so $\|y_i\| d^T v_i = \frac{1}{\sqrt{\mathfrak{g}_3}} \bar{w}^T B e_i$ holds for all $i \in V$. Hence

$$\sum_{i \in V} \|y_i\| \sqrt{w_i} d^T v_i = \frac{1}{\sqrt{\mathfrak{g}_3}} \sum_{i \in V} \bar{w}^T B e_i \sqrt{w_i} = \frac{1}{\sqrt{\mathfrak{g}_3}} \bar{w}^T B \bar{w} = \sqrt{\mathfrak{g}_3}.$$

Thus by the Cauchy-Schwarz inequality (0.1.26) and by (9.3.11)

$$\begin{aligned} \mathfrak{g}_3 &= \left(\sum_{i \in V} \|y_i\| \sqrt{w_i} d^T v_i \right)^2 \\ &\leq \left(\sum_{i \in V} \|y_i\|^2 \right) \left(\sum_{i \in V} w_i (d^T v_i)^2 \right) \\ &= (\text{tr}(B)) \left(\sum_{i \in V} w_i (d^T v_i)^2 \right) \\ &= \sum_{i \in V} w_i (d^T v_i)^2 \\ &\leq \mathfrak{g}_4(G, w). \end{aligned}$$

This proves (9.3.18).

Finally, we prove

$$(9.3.19) \quad \mathfrak{A}_4(G, w) \leq \mathfrak{A}(G, w) .$$

Choose an orthonormal representation $(v_i \mid i \in V)$, $v_i \in \mathbb{R}^N$, of \overline{G} and a vector $d \in \mathbb{R}^N$, $\|d\| = 1$, such that the maximum in the definition of $\mathfrak{A}_4(G, w)$ is achieved – see (9.3.11). We claim that the vector $((d^T v_i)^2 \mid i \in V)^T$ belongs to $\text{TH}(G)$. To see this, let $(u_i \mid i \in V)$, $u_i \in \mathbb{R}^N$, be any orthonormal representation of G and $c \in \mathbb{R}^N$ be any vector with unit length. We consider the matrices $u_i v_i^T \in \mathbb{R}^{N \times N}$. Note that these matrices are mutually orthogonal with respect to the inner product \bullet in $\mathbb{R}^{N \times N}$ and have unit length, that is

$$(u_i v_i^T) \bullet (u_j v_j^T) = (u_i^T u_j)(v_i^T v_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Similarly, the matrix cd^T has unit length with respect to this inner product. Hence

$$1 = (cd^T) \bullet (cd^T) \geq \sum_{i \in V} ((cd^T) \bullet (u_i v_i^T))^2 = \sum_{i \in V} (c^T u_i)^2 (d^T v_i)^2 .$$

This shows that the vector $((d^T v_i)^2 \mid i \in V)^T$ is in $\text{TH}(G)$, and so

$$\mathfrak{A}_4(G, w) = \sum_{i \in V} w_i (d^T v_i)^2 \leq \mathfrak{A}(G, w)$$

by the definition of $\mathfrak{A}(G, w)$ in (9.3.5).

The inequalities (9.3.13), (9.3.14), (9.3.17), (9.3.18), and (9.3.19) immediately imply Theorem (9.3.12). \square

(9.3.20) Remarks. (a) Since we have equality in all estimates (9.3.13), (9.3.14), (9.3.17), (9.3.18), and (9.3.19), it follows that the inequalities in their proofs must be tight, and this yields a number of further structural properties of optimal orthogonal representations and optimal matrices. In particular, equality in (9.3.14) implies that for every graph G and every weighting $w : V \rightarrow \mathbb{R}_+$ there exists an orthonormal representation $(u_i \mid i \in V)$, $u_i \in \mathbb{R}^N$, of G and a vector $c \in \mathbb{R}^N$, $\|c\| = 1$, such that $\mathfrak{A}(G, w)(c^T u_i)^2 = w_i$ for all $i \in V$.

(b) It is easy to see from the previous considerations – details will follow in Section 9.4 – that we have also obtained a lower bound on the weighted chromatic number of a graph, namely the following inequality holds

$$\omega(G, w) \leq \mathfrak{A}(\overline{G}, w) \leq \chi(G, w) .$$

In particular, we have

$$\omega(G) \leq \mathfrak{A}(\overline{G}, \mathbf{1}) \leq \chi(G) .$$

\square

The next lemma formulates a further property of orthonormal representations which will be needed in the sequel.

(9.3.21) Lemma. Let $(v_i \mid i \in V)$, $v_i \in \mathbb{R}^N$, be an orthonormal representation of \overline{G} and $d \in \mathbb{R}^N$ with $\|d\| = 1$. Let $w : V \rightarrow \mathbb{R}_+$ be any weighting of the nodes of G , and assume that

$$\sum_{i \in V} (d^T v_i)^2 w_i = \mathfrak{g}(G, w).$$

Then

$$\sum_{i \in V} w_i (d^T v_i) v_i = \mathfrak{g}(G, w) d.$$

Proof. By (9.3.11) and (9.3.12) we have

$$\sum_{i \in V} w_i (y^T v_i)^2 \leq \mathfrak{g}(G, w)$$

for all unit vectors $y \in \mathbb{R}^N$, and so by assumption, the left hand side is maximized by $y = d$. But the left hand side can be written as a quadratic form in y as follows:

$$\sum_{i \in V} w_i (y^T v_i)^2 = y^T \left(\sum_{i \in V} w_i v_i v_i^T \right) y,$$

and so its maximum over all vectors y of unit length is attained when y is an eigenvector of $\sum_{i \in V} w_i v_i v_i^T$, and the maximum value is the corresponding eigenvalue. Hence d is an eigenvector of this matrix associated with the eigenvalue $\mathfrak{g}(G, w)$. This implies

$$\sum_{i \in V} w_i (d^T v_i) v_i = \left(\sum_{i \in V} w_i v_i v_i^T \right) d = \mathfrak{g}(G, w) d.$$

□

Let us mention some corollaries of Theorem (9.3.12). We will first show that there are a number of further ways to describe the convex set $\text{TH}(G)$ defined in (9.3.3).

(9.3.22) Corollary. Each of the following expressions defines $\text{TH}(G)$:

- (a) $\text{TH}(G) = \{ x \in \mathbb{R}_+^V \mid \mathfrak{g}(\overline{G}, x) \leq 1 \}$;
- (b) $\text{TH}(G) = \{ x \in \mathbb{R}_+^V \mid x^T y \leq \mathfrak{g}(G, y) \text{ for all } y \in \mathbb{R}_+^V \}$;
- (c) $\text{TH}(G) = \{ ((d^T v_i)^2 \mid i \in V) \mid \text{where } (v_i \mid i \in V), v_i \in \mathbb{R}^N, \text{ is an orthonormal representation of } \overline{G} \text{ and } d \in \mathbb{R}^N \text{ satisfies } \|d\| = 1 \}$.

Proof. (a) follows if, using that $\mathfrak{g} = \mathfrak{g}_4$ by Theorem (9.3.12), we substitute

$$\mathfrak{g}(\overline{G}, x) = \max \sum_{i \in V} (c^T u_i)^2 x_i,$$

where $(u_i \mid i \in V)$, $u_i \in \mathbb{R}^N$, is an orthonormal representation of G and $c \in \mathbb{R}^N$, $\|c\| = 1$.

(b) Each $x \in \text{TH}(G)$ satisfies $x^T y \leq \vartheta(G, y)$ by the definition of $\vartheta(G, y)$ in (9.3.5). On the other hand, setting for each orthonormal representation $(u_i \mid i \in V)$ and each vector c with $\|c\| = 1$, $y_i := (c^T u_i)^2$ and $y := (y_i \mid i \in V)$, we conclude from (9.3.6) and (9.3.12) that $\vartheta(G, y) \leq 1$. And thus the inequalities (9.3.2) are implied by the inequalities $x^T y \leq \vartheta(G, y)$ for all $y \in \mathbb{R}_+^V$.

(c) In the proof of inequality (9.3.19) we have already shown that $((d^T v_i)^2 \mid i \in V) \in \text{TH}(G)$ for each orthonormal representation (v_i) of \bar{G} and each unit vector d . Conversely, let $x \in \text{TH}(G)$. Then by (a) $t := \vartheta(\bar{G}, x) \leq 1$. By Remark (9.3.20) (a), \bar{G} has an orthonormal representation $(v_i \mid i \in V)$, $v_i \in \mathbb{R}^N$, and there is a vector $d \in \mathbb{R}^N$, $\|d\| = 1$ such that $t(d^T v_i)^2 = x_i$ for all $i \in V$. Let d_1 be a vector orthogonal to d , and all v_i , $i \in V$ (in any linear space containing \mathbb{R}^N) with $\|d_1\| = 1$, and set $d_0 := \sqrt{t}d + \sqrt{1-t}d_1$. Then $\|d_0\| = 1$ and $(d_0^T v_i)^2 = t(d^T v_i)^2 = x_i$ for all $i \in V$. This proves (c). \square

Combining definition (9.3.3) of $\text{TH}(G)$ with (9.3.22) (c) we get as an immediate consequence:

(9.3.23) Corollary. *The antiblocker of $\text{TH}(G)$ is equal to $\text{TH}(\bar{G})$.* \square

We have already mentioned that $\text{TH}(G)$ is not a polytope in general. We will now characterize its facets.

(9.3.24) Theorem. *If an inequality defines a facet of $\text{TH}(G)$, then it is a positive multiple of one of the nonnegativity constraints (9.1.1) or of one of the clique constraints (9.2.1).*

Proof. Suppose that $a^T x \leq \alpha$ defines a facet of $\text{TH}(G)$, and let z be a point in the relative interior of $F := \{x \in \text{TH}(G) \mid a^T x = \alpha\}$. Then either $z_i = 0$ for some i , or $\vartheta(\bar{G}, z) = 1$ by (9.3.22) (a). In the first case $a^T x \leq \alpha$ is trivially equivalent to $x_i \geq 0$. So suppose that $\vartheta(\bar{G}, z) = 1$. Since $\vartheta = \vartheta_4$ by Theorem (9.3.12), there exists an orthonormal representation (u_i) of G and a unit vector c such that

$$\sum_{i \in V} z_i (c^T u_i)^2 = 1.$$

Since by definition (9.3.3) the orthonormal representation constraint

$$(9.3.25) \quad \sum_{i \in V} x_i (c^T u_i)^2 \leq 1$$

is satisfied by all $x \in \text{TH}(G)$, it follows that (9.3.25) must be equivalent with $a^T x \leq \alpha$, and hence we may assume that $\alpha = 1$ and $a_i = (c^T u_i)^2$. We also see that

$$(9.3.26) \quad \sum_{i \in V} x_i (c^T u_i)^2 = 1 = \vartheta(\bar{G}, x)$$

for all $x \in F$, and thus by Lemma (9.3.21)

$$\sum_{i \in V} x_i (c^T u_i) u_i = c$$

holds for all $x \in F$, in other words,

$$\sum_{i \in V} x_i (c^T u_i) (u_i)_j = c_j$$

for all $j \in V$. Since F is $(n - 1)$ -dimensional, these equations must follow from the equation (9.3.26), and hence $c_j (c^T u_i)^2 = (c^T u_i) (u_i)_j$ for all $i, j \in V$. Hence for any $i \in V$ such that $c^T u_i \neq 0$ we have $(c^T u_i) c = u_i$. Since $\|u_i\| = \|c\| = 1$, this yields that $u_i = \pm c$. Clearly, we may assume that $u_i = c$.

So we see that for each i , either $c^T u_i = 0$ or $u_i = c$. Set $Q := \{i \in V \mid u_i = c\}$. Then Q is a complete subgraph, since for any two nodes $i, j \in Q$, $u_i^T u_j = c^T c = 1 \neq 0$. So (9.3.25) is just the clique inequality

$$\sum_{i \in Q} x_i \leq 1,$$

which proves the theorem. □

(9.3.27) Corollary. *TH(G) is a polytope if and only if G is a perfect graph.*

Proof. If G is perfect then by definition $\text{STAB}(G) = \text{QSTAB}(G)$. Thus by (9.3.4) $\text{STAB}(G) = \text{TH}(G)$, which implies that $\text{TH}(G)$ is a polytope.

Conversely, suppose that $\text{TH}(G)$ is a polytope. Then $\text{TH}(G)$ is the solution set of all inequalities determining a facet of $\text{TH}(G)$. By Theorem (9.3.24), all these inequalities are either nonnegativity constraints (9.1.1) or clique constraints (9.2.1), and so $\text{TH}(G) = \text{QSTAB}(G)$. By Corollary (9.3.23) $\text{TH}(\overline{G}) = \text{abl}(\text{TH}(G))$, and hence $\text{TH}(\overline{G})$ is also a polytope. Using the same arguments as above we conclude that $\text{TH}(\overline{G}) = \text{QSTAB}(\overline{G})$. But now $\text{STAB}(G) = \text{abl}(\text{QSTAB}(\overline{G})) = \text{abl}(\text{TH}(\overline{G})) = \text{TH}(G) = \text{QSTAB}(G)$, and thus G is a perfect graph. □

Two immediate corollaries are the following:

(9.3.28) Corollary. *TH(G) = STAB(G) if and only if G is perfect.* □

(9.3.29) Corollary. *TH(G) = QSTAB(G) if and only if G is perfect.* □

The theorems and corollaries above show that in spite of its complicated definition, $\text{TH}(G)$ does have nice and useful properties. For our purposes the following property of $\text{TH}(G)$ is the most important.

(9.3.30) Theorem. *The weak optimization problem for $\text{TH}(G)$ is solvable in polynomial time.*

It is necessary to restrict our attention to weak optimization, because $\text{TH}(G)$ is not a polyhedron in general and so the results of Chapter 6 do not apply. To see an example, let G be the pentagon C_5 and let $x_0 \in \mathbb{R}^5$ be the vector maximizing the linear objective function $\mathbf{1} = (1, 1, 1, 1, 1)^T$ over $\text{TH}(C_5)$. By symmetry, we may assume that $x_0 = (t, t, t, t, t)^T$ for some $t > 0$. So

$$\mathbf{1}^T x \leq \mathbf{1}^T x_0 = 5t$$

is valid for all $x \in \text{TH}(G)$ and so the vector $\frac{1}{5t}\mathbf{1}$ belongs to the antiblocker of $\text{TH}(C_5)$. But C_5 is self-complementary and hence $\frac{1}{5t}\mathbf{1} \in \text{TH}(C_5)$ by Corollary (9.3.23). Thus $\mathbf{1}^T(\frac{1}{5t}\mathbf{1}) \leq 5t$, whence $t \geq \frac{1}{\sqrt{5}}$. As remarked before, the orthonormal representation constraint derived from the orthonormal representation of C_5 shown in Figure 9.3 reads $\sum_{i \in C_5} x_i \leq \sqrt{5}$. This implies $t = \frac{1}{\sqrt{5}}$. So neither the maximizing vector nor the maximum value are rational.

Another preliminary remark : it would not do much good to apply the optimization-separation equivalence at this point. In fact, the separation problem for $\text{TH}(G)$ is equivalent to the violation problem for its antiblocker; but this is just $\text{TH}(\overline{G})$, and so all we can achieve by this is to reduce the optimization (or violation) problem for $\text{TH}(G)$ to the same problem for the complementary graph.

Proof of (9.3.30). Since $\text{TH}(G)$ is closed and convex, and it is trivial to find inscribed and circumscribed balls, it suffices to solve the weak validity problem.

So let $c^T x \leq \gamma$ be any inequality, $c \in \mathbb{Q}^V$, $\gamma \in \mathbb{Q}$. If $c_i \leq 0$ for some $i \in V$ then clearly it suffices to check whether, dropping the i -th coordinate, the remaining inequality is valid for $\text{TH}(G - i)$. So we may assume that $c > 0$.

By the definition of \mathfrak{D} , our task is done if we can compute $\mathfrak{D}(G, c)$ in polynomial time (in the sense of computing real numbers in polynomial time). For this, we can use formula (9.3.10):

$$\mathfrak{D}(G, c) = \mathfrak{D}_3(G, c) = \max\{\bar{c}^T B \bar{c} \mid B \in \mathfrak{D} \cap \mathcal{M} \text{ and } \text{tr}(B) = 1\},$$

where \mathfrak{D} and \mathcal{M} are the sets defined in (9.3.7). To see that this maximum is indeed computable in polynomial time, we apply Yudin and Nemirovskii's theorem (4.3.2). For this, consider the affine subspace

$$\mathcal{M}_1 := \{B \in \mathcal{M} \mid \text{tr}(B) = 1\}$$

and the set

$$\mathcal{K} := \mathcal{M}_1 \cap \mathfrak{D}.$$

So we have $\mathfrak{D}(G, c) = \max\{\bar{c}^T B \bar{c} \mid B \in \mathcal{K}\}$.

Claim 1. \mathcal{K} is convex.

This follows since both \mathcal{M}_1 and \mathfrak{D} are convex.

Claim 2. $\mathcal{K} \subseteq S(0, n)$.

Indeed, let $B = (b_{ij}) \in \mathcal{X}$. Since B is positive semidefinite, $b_{ii} \geq 0$. Since $\text{tr} B = 1$, we also have $b_{ii} \leq 1$. Using that B is positive semidefinite again, we have that $b_{ii}b_{jj} - b_{ij}^2 \geq 0$, and hence $|b_{ij}| \leq \sqrt{b_{ii}b_{jj}} \leq 1$. Hence $\|B\|_{\max} \leq n$, and so by (0.1.24) $B \in S(0, n)$.

Claim 3. $S(\frac{1}{n}I, \frac{1}{n}) \cap \mathcal{M}_1 \subseteq \mathcal{X}$.

For, let $B \in \mathcal{M}_1$ be any symmetric matrix in $\mathbb{R}^{V \times V}$ such that $\|\frac{1}{n}I - B\|_2 \leq \frac{1}{n}$. Then by (0.1.25) the largest eigenvalue of $\frac{1}{n}I - B$ is at most $\frac{1}{n}$. So the smallest eigenvalue of B is at least 0. Thus by (0.1.4) $B \in \mathcal{D}$, and so $B \in \mathcal{D} \cap \mathcal{M}_1 = \mathcal{X}$.

The first three claims show that \mathcal{X} is a centered, well-bounded, convex body in \mathcal{M}_1 . In what follows, we shall work in the space \mathcal{M}_1 . (To be precise, we project \mathcal{M}_1 (and \mathcal{X}) on the space \mathbb{R}^S where $S = \{(i, j) \mid ij \notin E, i \leq j, (i, j) \neq (n, n)\}$.)

Claim 4. The weak membership problem for \mathcal{X} is solvable in polynomial time.

All we have to do is to check whether a given symmetric matrix B is positive semidefinite. There are various characterizations of positive semidefiniteness – see (0.1.4) – which can be used for the design of an efficient proof of this property.

For instance, polynomial time algorithms can be obtained from Gaussian elimination and Cholesky decomposition. In Gaussian elimination we allow pivots on the main diagonal only; if the rank of the matrix is found and only positive pivots have been carried out, then the matrix is positive semidefinite. Cholesky decomposition can be used in a similar way. A further method is to compute the smallest eigenvalue λ_n ; if λ_n is nonnegative then the matrix is positive semidefinite. This algorithm may be fast in practice but it is not so easy to implement it in polynomial time.

Let us describe in detail the algorithm based on Gaussian elimination – see Section 1.4. To check positive semidefiniteness we use criterion (0.1.4) (iv). So let $B = (b_{ij})$ be a symmetric rational $n \times n$ -matrix. If $b_{11} < 0$ then B is not positive semidefinite. If $b_{11} = 0$ but $b_{1i} \neq 0$ for some $2 \leq i \leq n$ then B is not positive semidefinite (in fact, its principal submatrix $\begin{pmatrix} b_{11} & b_{1i} \\ b_{1i} & b_{ii} \end{pmatrix}$ has determinant $-b_{1i}^2 < 0$). If $b_{11} = 0$ and also $b_{1i} = 0$ for all i , then we can drop the first row and column and restrict our attention to the remaining $(n - 1) \times (n - 1)$ -matrix. If $b_{11} > 0$ then pivot on b_{11} , i. e., consider the matrix $B' = (b'_{ij})_{i,j=2}^n$ where $b'_{ij} := b_{ij} - \frac{b_{i1}b_{1j}}{b_{11}}$. Then B' is positive semidefinite if and only if B is, and so again we have reduced our task to an $(n - 1) \times (n - 1)$ problem.

It has to be checked, however, that the entries of the smaller matrices obtained this way do not grow too large, i. e., their encoding lengths remain bounded by the encoding length of B . This is easy to see, however, since these entries are subdeterminants of the original matrix B – see the proof of Theorem (1.4.8). \square

(9.3.31) Remark. One could also use formula (9.3.9)

$$\mathfrak{G}(G, w) = \min\{\Lambda(A + W) \mid A \in \mathcal{M}^\perp\}$$

to obtain $\mathfrak{G}(G, w)$. Namely, $\Lambda(A + W)$ is a convex function of A , and A ranges through a linear space. So we obtain an unconstrained convex function minimization problem. Using standard linear algebra, one can design an evaluation oracle

for $\Lambda(A+W)$ and also derive a bound R on the Frobenius norm of the minimizing matrix. Then Theorem (4.3.13) can be applied to compute $\mathfrak{G}(G, w)$. \square

Corollary (9.3.27), Theorem (9.3.30), and Theorem (6.3.2) immediately give :

(9.3.32) Corollary. *The weighted stable set problem for perfect graphs is solvable in polynomial time.* \square

Trivially, we also obtain the following:

(9.3.33) Corollary. *The weighted clique problem for perfect graphs is solvable in polynomial time.* \square

*9.4 Coloring Perfect Graphs

We shall now describe how the results of Section 9.3 can be used to solve further optimization problems for perfect graphs in polynomial time. Recall the following two problems.

(9.4.1) Minimum Weighted Coloring Problem. *Given a graph $G = (V, E)$ with node weights $w : V \rightarrow \mathbb{Z}_+$, find stable sets $S_1, \dots, S_k \subseteq V$ and nonnegative integers $\lambda_1, \dots, \lambda_k$ such that for each node $v \in V$ the sum $\sum_{i, v \in S_i} \lambda_i$ is not smaller than $w(v)$ and such that $\lambda_1 + \dots + \lambda_k$ is as small as possible.*

(9.4.2) Minimum Weighted Clique Covering Problem. *Given a graph $G = (V, E)$ with node weights $w : V \rightarrow \mathbb{Z}_+$, find cliques $Q_1, \dots, Q_k \subseteq V$ and nonnegative integers $\lambda_1, \dots, \lambda_k$ such that for each node $v \in V$ the sum $\sum_{i, v \in Q_i} \lambda_i$ is not smaller than $w(v)$ and such that $\lambda_1 + \dots + \lambda_k$ is as small as possible.*

As before, we denote the optimum value of (9.4.1) by $\chi(G, w)$ and the optimum value of (9.4.2) by $\bar{\chi}(G, w)$. Clearly, the coloring problem (9.4.1) for G is nothing but the clique covering problem (9.4.2) for the complementary graph \bar{G} , and vice versa. Both problems are \mathcal{NP} -hard for general graphs. We will now show that they are solvable in polynomial time for perfect graphs. It suffices to discuss problem (9.4.2).

(9.4.3) Theorem. *There is an algorithm that solves the minimum weighted clique covering problem for perfect graphs in polynomial time.*

Proof. An integer programming formulation of (9.4.2) can be given as follows.

$$(9.4.4) \quad \min \sum_{Q \in \mathcal{V} \text{ clique}} \lambda_Q$$

$$(i) \quad \sum_{Q \text{ clique}, Q \ni v} \lambda_Q \geq w(v) \quad \text{for all } v \in V,$$

- (ii) $\lambda_Q \geq 0$ for all cliques $Q \subseteq V$,
- (iii) λ_Q integral for all cliques $Q \subseteq V$.

The linear program (9.4.4) without the integrality stipulation (iii) is the dual of the linear program

$$(9.4.5) \quad \max w^T x, \quad x \in \text{QSTAB}(G),$$

where $\text{QSTAB}(G)$ is the polytope defined by the nonnegativity and clique constraints – see (9.2.7). (By Theorem (9.2.9), problem (9.4.5) is \mathcal{NP} -hard in general, hence even the LP-relaxation of (9.4.4) is \mathcal{NP} -hard.) By (9.3.28) and (9.3.29) we have $\text{QSTAB}(G) = \text{STAB}(G) = \text{TH}(G)$ if and only if G is perfect, and thus by (9.3.32), an optimum solution of (9.4.5) can be found in polynomial time if G is a perfect graph.

Therefore, our general results of Chapter 6 imply that we can find an optimum solution of the linear program dual to (9.4.5) in polynomial time, and thus an optimum solution of (9.4.4) without the integrality condition (iii) can be computed in polynomial time, in case G is a perfect graph. But by Theorem (9.2.4), if G is a perfect graph, the integrality condition (iii) of (9.4.4) is superfluous, i. e., the linear program (9.4.4) (i), (ii) always has an optimum solution which is integral. Our method, however, for finding an optimum dual solution is only guaranteed to find a basic optimum solution, and since (9.4.4) without (iii) may have nonintegral basic optimum solutions (in addition to the integral ones) we are not sure that the general techniques solve (9.4.4). But in this special case it is possible to construct an integral optimum solution from a fractional one. This goes as follows.

By finding a basic optimal solution of the LP-dual of (9.4.5) we obtain a basic optimum solution of (9.4.4) without (iii). It is a trivial matter to modify this solution such that it is optimal for

$$(9.4.6) \quad \min \sum_{Q \subseteq V \text{ clique}} \lambda_Q$$

- (i) $\sum_{Q \text{ clique}, Q \ni v} \lambda_Q = w(v)$ for all $v \in V$,
- (ii) $\lambda_Q \geq 0$ for all cliques $Q \subseteq V$.

Let Q_1 be any clique such that $\lambda_{Q_1} > 0$ in this optimum solution. Because of condition (i) of (9.4.6) we have $w(q) > 0$ for all $q \in Q_1$. Moreover by complementary slackness, Q_1 has a nonempty intersection with all stable sets $S \subseteq V$ satisfying $w(S) = \alpha(G, w)$.

We define $w'(v) := w(v)$ for all $v \in V \setminus Q_1$, and using the algorithm of Corollary (9.3.32), we compute $\alpha(G - Q_1, w')$. We set

$$\mu_{Q_1} := \min\{\alpha(G, w) - \alpha(G - Q_1, w'), \min\{w(q) \mid q \in Q_1\}\}.$$

Clearly, μ_{Q_1} is a positive integer. Consider the new weighting

$$(9.4.7) \quad w_1 := w - \mu_{Q_1} \chi^{Q_1}.$$

By our choice of μ_{Q_1} we have $w_1 \geq 0$, and our construction yields

$$\alpha(G, w_1) = \alpha(G, w) - \mu_{Q_1}.$$

To see that this equation holds, observe first that $\alpha(G, w_1) \geq \alpha(G, w) - \mu_{Q_1}$, since there exists a stable set S with $\alpha(G, w) = w(S) \leq w_1(S) + \mu_{Q_1} \leq \alpha(G, w_1) + \mu_{Q_1}$. To show the converse inequality, let S be any stable set satisfying $w_1(S) = \alpha(G, w_1)$. If $S \cap Q_1 = \emptyset$ then $\alpha(G, w_1) = w_1(S) = w'(S) \leq \alpha(G - Q_1, w') \leq \alpha(G, w) - \mu_{Q_1}$. If $S \cap Q_1 \neq \emptyset$ then $\alpha(G, w_1) = w_1(S) \leq w(S) - \mu_{Q_1} \leq \alpha(G, w) - \mu_{Q_1}$.

If $w_1 \neq 0$ we repeat this procedure, with w replaced by w_1 .

This algorithm ends up with a list of cliques Q_1, \dots, Q_k and of positive integers $\mu_{Q_1}, \dots, \mu_{Q_k}$ such that

$$\begin{aligned} \mu_{Q_1} \chi^{Q_1} + \dots + \mu_{Q_k} \chi^{Q_k} &= w \\ \mu_{Q_1} + \dots + \mu_{Q_k} &= \alpha(G, w). \end{aligned}$$

Thus we have found an integral solution of (9.4.6) and hence a solution of (9.4.4). Clearly, each step of the iteration described above can be executed in polynomial time. To finish the running time analysis of our algorithm we prove the following:

Claim. The vectors $\chi^{Q_1}, \dots, \chi^{Q_k}$ are linearly independent.

We show this by proving that for each i , $1 \leq i \leq k$, there exists a vector orthogonal to $\chi^{Q_{i+1}}, \dots, \chi^{Q_k}$ but not to χ^{Q_i} . Let us denote the vector w_1 defined in (9.4.7) in the i -th iteration of the algorithm by w_i , $i = 1, \dots, k$. By the definition of μ_{Q_i} there are two cases. Either there is a node $q \in Q_i$ such that $w_i(q) = 0$ or there exists a stable set $S \subseteq V(G) \setminus Q_i$ such that $w_i(S) = \alpha(G, w_i)$. In the first case, $w_j(q) = 0$ for all $i \leq j \leq k - 1$ and hence $q \notin Q_{j+1}$. So $\chi^{(q)}$ is orthogonal to $\chi^{Q_{i+1}}, \dots, \chi^{Q_k}$ but not to χ^{Q_i} . In the second case observe that $-S$ is a maximum weight stable set of G not only with respect to the weighting w_i but also with respect to w_{i+1}, \dots, w_{k-1} . Hence by complementary slackness $|S \cap Q_{j+1}| = 1$ for $j = i, \dots, k - 1$. If S_0 is a maximum weight stable set of G with respect to the initial weighting w then $|S_0 \cap Q_j| = 1$ for all j , and hence $\chi^{S_0} - \chi^S$ is a vector orthogonal to $\chi^{Q_{i+1}}, \dots, \chi^{Q_k}$ but not to χ^{Q_i} . This proves the claim.

It follows from our claim that $k \leq n$, and hence the algorithm terminates in polynomial time. Moreover, the claim shows that $\mu_{Q_1}, \dots, \mu_{Q_k}$ yields a basic optimum dual solution of (9.4.5). \square

Since the coloring problem (9.4.1) for a graph G is the clique covering problem (9.4.2) for the complementary graph \bar{G} , and since a graph is perfect if and only if its complement is perfect we obtain :

(9.4.8) Corollary. *There is an algorithm that solves the minimum weighted coloring problem for perfect graphs in polynomial time.* \square

*9.5 More Algorithmic Results on Stable Sets

In this last section we survey some further classes of graphs for which the stable set problem can be solved in polynomial time.

Given any class \mathcal{L} of valid inequalities for $\text{STAB}(G)$, we can consider the polyhedron

$$(9.5.1) \quad \mathcal{L} \text{ STAB}(G) := \{ x \in \mathbb{R}_+^V \mid x \text{ satisfies } \mathcal{L} \},$$

and ask ourselves the question whether the optimization problem for $\mathcal{L} \text{ STAB}(G)$ can be solved in polynomial time. By the general results of Chapters 4 and 6, this is equivalent to checking whether or not a given vector $y \in \mathbb{R}_+^V$ belongs to $\mathcal{L} \text{ STAB}(G)$, i. e., whether or not y satisfies all the inequalities in \mathcal{L} . If this problem can be solved in polynomial time, then the stable set problem can be solved in polynomial time for all graphs G such that $\mathcal{L} \text{ STAB}(G) = \text{STAB}(G)$. Let us call a graph G **\mathcal{L} -perfect** if $\mathcal{L} \text{ STAB}(G) = \text{STAB}(G)$.

The significance of such a result depends on whether or not there are any interesting examples of \mathcal{L} -perfect graphs. In Sections 9.1, 9.2, and 9.3 we have studied two classes \mathcal{L} of valid inequalities for which such examples do exist. Here we shall mention some further classes for which the membership problem for $\mathcal{L} \text{ STAB}(G)$ can be solved in polynomial time, even though we do not know really interesting classes of \mathcal{L} -perfect graphs.

As a first remark, note that if the membership problem is solvable for $\mathcal{L}_1 \text{ STAB}(G)$ and $\mathcal{L}_2 \text{ STAB}(G)$ in polynomial time then it is also solvable for $\mathcal{L}_1 \text{ STAB}(G) \cap \mathcal{L}_2 \text{ STAB}(G) = (\mathcal{L}_1 \cup \mathcal{L}_2) \text{ STAB}(G)$. So we may take the nonnegativity constraints (9.1.1) and the odd circuit constraints (9.1.4) together with the orthonormal representation constraints (9.3.2) and obtain a convex set containing $\text{STAB}(G)$ for which the optimization problem can be solved in polynomial time. The corresponding “perfect” graphs are defined by

$$(9.5.2) \quad \text{TH}(G) \cap \text{CSTAB}(G) = \text{STAB}(G) .$$

It follows from Theorem (9.3.24) that this condition is equivalent to the following:

$$(9.5.3) \quad \text{QSTAB}(G) \cap \text{CSTAB}(G) = \text{STAB}(G) .$$

Graphs satisfying (9.5.3) are called **h -perfect**; these are those graphs for which the nonnegativity, clique, and odd circuit constraints suffice to describe $\text{STAB}(G)$. Since every h -perfect graph also satisfies (9.5.2), we obtain the following.

(9.5.4) Corollary. *The stable set problem for h -perfect graphs can be solved in polynomial time.* □

We do not know any interesting classes of h -perfect graphs that are neither perfect, nor t -perfect, nor combinations of these. But, of course, there are such graphs. For instance, the graph in Figure 9.4 is h -perfect but neither perfect nor t -perfect.

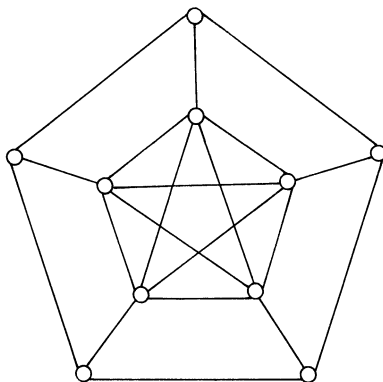


Figure 9.4

The **wheel constraints** for $\text{STAB}(G)$ are determined by the **odd wheels** in G , i. e., induced subgraphs $G[W]$ of G containing a node w (the **center**) such that w is adjacent to all other nodes of $G[W]$ and $G[W] - w$ is an odd circuit. Figure 9.5 shows a wheel, where $G[W] - w$ is a 7-circuit.

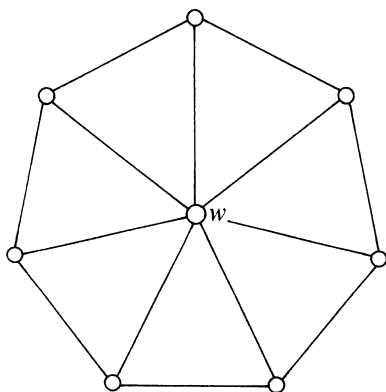


Figure 9.5

Given such a subgraph $G[W]$, where $|W|$ is even, we can write up the following **odd wheel constraint**:

$$(9.5.5) \quad x(W \setminus \{w\}) + \frac{|W|-2}{2}x_w \leq \frac{|W|-2}{2}.$$

Note that (9.5.5) is stronger than the odd circuit constraint determined by $G[W] - w$. Let \mathcal{W} denote the class of constraints (9.1.1), (9.1.2), (9.1.4), (9.5.5). Then we have

(9.5.6) Theorem. *The strong separation (and the strong optimization) problem for \mathcal{W} STAB(G) can be solved in polynomial time.*

Proof. Let $y \in \mathbb{Q}^V$. We know that we can check in polynomial time whether y satisfies the nonnegativity constraints (9.1.1), the edge constraints (9.1.2), and the odd circuit constraints (9.1.4). Suppose that we find that it does. Then define, for each node $w \in V$, a weighting of the edges spanned by the set $\Gamma(w)$ of neighbors of w , by

$$l_w(uv) := 1 - y_u - y_v - y_w \quad \text{for all } uv \in E \text{ with } u, v \in \Gamma(w).$$

Since y satisfies the odd circuit constraints, we know that $l_w(uv) \geq 0$. So for each node $w \in V$, we can find, by the algorithm solving problem (8.3.6), an odd circuit of $(\Gamma(w), E(\Gamma(w)))$ with smallest l_w -length. Suppose that we find this way an odd circuit (u_0, \dots, u_{2k}) with l_w -length less than $1 - y(w)$. This means

$$\sum_{i=0}^{2k} (1 - y(u_i) - y(u_{i+1}) - y(w)) < 1 - y(w)$$

and so

$$\sum_{i=0}^{2k} y(u_i) + ky(w) > k.$$

Hence, y violates an odd wheel constraint. Conversely, if for every $w \in V$ every odd circuit in $\Gamma(w)$ has l_w -length at least $1 - y(w)$, then clearly $y \in \mathcal{W}$ STAB(G). \square

It follows that the stable set problem can be solved in polynomial time for \mathcal{W} -perfect graphs. Unfortunately, we are not aware of any interesting examples of such graphs.

There are further classes of inequalities known that are valid for STAB(G) or define facets – see for instance PADBERG (1973), NEMHAUSER and TROTTER (1974), or – for a survey – BALAS and PADBERG (1975). For most of these classes we do not know whether or not the associated separation problem can be solve in polynomial time. An example of such a class is the class of **odd antihole inequalities**. These are inequalities of the form $x(W) \leq 2$ where the subgraph of G induced by W is the complement of a chordless odd circuit of length at least five.

The papers NEMHAUSER and TROTTER (1974) and PADBERG (1975) describe general methods – so-called lifting techniques – with which, for each induced subgraph G' of G , valid (or facet defining) inequalities for STAB(G') can be turned into valid (or facet defining) inequalities for STAB(G). (These methods apply to integral polyhedra in general.) In fact, the odd wheel constraints can be obtained from the odd circuit constraints with this lifting technique, and we are able to modify the trick described in the proof of Theorem (9.5.6) to solve the separation problem for further classes of lifted inequalities. Similarly, we can solve the separation problem for the separation problem for \mathcal{L} STAB(G) if \mathcal{L} consists of

constraints derived from odd subdivisions of K_4 , or, more generally, from odd subdivisions of wheels. But a general theory of such classes of inequalities is not known. It is also an open question whether the more involved linear algebraic methods of Section 9.3 can be pushed any further.

We conclude with some classes of graphs for which the stable set problem is polynomially solvable, although not directly by polyhedral methods.

Since the stable set problem for the line graph of a graph is trivially equivalent to the matching problem for the original graph, it is solvable in polynomial time (see Section 7.3). MINTY (1980) and SBIHI (1978, 1980) extended this result to the stable set problem of claw-free graphs, i. e., graphs not containing a 3-star as an induced subgraph. (It is easy to see that every line graph is claw-free, but not vice versa.) Minty also solved the weighted version in polynomial time. However, in spite of considerable efforts, no decent system of inequalities describing $\text{STAB}(G)$ for claw-free graphs is known – cf. GILES and TROTTER (1981).

Let us remark that for some interesting classes of graphs, the number of all maximal stable sets is polynomial in $|V|$, and all such sets can be actually listed in polynomial time :

complements of triangulated graphs,
complements of bipartite graphs, and
complements of line-graphs

are some examples. Now, if $G = (V, E)$ is a graph whose nodes can be partitioned into two classes V_1 and V_2 such that both $G[V_1]$ and $G[V_2]$ belong to one of the classes above, then the stable set problem for G is easily solved in polynomial time, by solving it for all bipartite graphs $G[S_1 \cup S_2]$, where S_i is a maximal stable set in $G[V_i]$.

One of the fastest branch and bound methods for the cardinality version of the stable set (resp. clique) problem in arbitrary graphs known to date is based on observations of this type. The algorithm is due to BALAS and YU (1984). It iteratively generates maximal triangulated induced subgraphs or complements of these and heavily exploits the fact that maximum stable sets in such graphs can be computed in time linear in $|V| + |E|$. The algorithm terminates in polynomial time for instance for complements of triangulated graphs, but also for some classes of graphs that include some imperfect graphs. BALAS, CHVÁTAL and NEŠETŘIL (1985) have modified and extended these ideas to obtain algorithms that solve the weighted stable set problem for several further classes of graphs in polynomial time. The common feature of these classes of graphs is that every stable set of any of these graphs is contained in some member of a polynomial-sized collection of induced subgraphs that are bipartite.

Hsu, IKURA and NEMHAUSER (1981) showed that for each fixed k the weighted stable set problem can be solved in polynomial time by enumerative methods if the longest odd cycle in the graph is bounded by k .

It follows from the results of Chapter 6 that, for each of the classes of graphs mentioned above, the facets of $\text{STAB}(G)$ can in a sense be described : we can decide in polynomial time whether or not a given inequality is valid for $\text{STAB}(G)$, and also whether or not it determines a facet of $\text{STAB}(G)$. Moreover, we can

represent every valid inequality as a nonnegative combination of facet defining inequalities of $\text{STAB}(G)$. We can also find, for any vector not in $\text{STAB}(G)$, a facet separating it from $\text{STAB}(G)$ in polynomial time. But of course, it would be interesting to find a direct description of $\text{STAB}(G)$ by linear inequalities and then use this to solve the stable set problem. For none of the three classes of graphs mentioned above, such a description of $\text{STAB}(G)$ is known.