Optimization **Summer 2016**

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1 Recap

1.1 Simplex algorithm

Let P be a polyhedron that defines feasible space and c be a vector of objective function. The simplex algorithm takes the following steps:

1. Start at some vertice of P
2. Move to the next vertice, that would improve objective function
3. Repeat step 2 until objective function cannot be improved anymore
4. Obtained vertice is the optimal solution

There are several ways to describe vertices mathematically:

- **Vertex**. There exists an objective function such that point v is the unique optimal solution
- **Extreme point**. The point of polyhedron P cannot be expressed as a convex combination of two other points in P.
- **Basic feasible solution**. There exist n linearly independent constraints that are tight in this point.

2 Optimal solution at vertices

2.1 When polyhedron has vertices

Applying the simplex algorithm we want it to finish in finite time. Therefore polyhedron P should not have infinite amount of vertexes. We claim that there exists an optimal solution, which is a vertice, apart from the following exceptions:

- P is unbounded
- there are no feasible solutions
- P does not have a vertex

We define a property of polyhedron, that detects whether a polyhedron has no vertexes.
DEFINITION A polyhedron P has no vertexes if it contains a line.
DEFINITION P contains a line if there exists a point $x \in P$ and vector d denoting direction, such that every point of the form $x + \lambda d \in P$.

Theorem The polyhedron $P = \{x \in \mathbb{R}^n | Ax \geq b\} \neq \varnothing$. The following statements are equal:

a) P has at least one extreme point

b) P does not contain a line

c) there are no n linearly independent constraints

Proof: $b) \Rightarrow a)$ Let's construct a polyhedron. Then we start in some point and move in direction of boundary of P considering we cannot have infinite amount of steps. We know that we are on the edge if some constraints are tight.
Let point $x \in P$, and define set of indexes $I = \{i | a_i^T x = b_i\}$ and set of rows, for which constraints are tight $S = \{a_i | i \in I\}$. If amount of rows in S is equal to n, then x is basic feasible solution, hence (according to the Theorem from Lecture 5) x is an extreme point.
Assume it's not the case. Then a_i lie in the subspace of $\mathbb{R}^n \Rightarrow \exists d \neq 0$ such that $a_i^T d = 0 \forall i \in I$. d is a direction in which we want to move. Now we are interested in points of a form $x + \lambda d$. As we move in direction of boundary, after some finite λ some constraint has to become tight (was not tight before). Hence $\exists \lambda^* > 0$ and $j \notin I$ such that $a_j^T (x + \lambda^* d) = b_i$. If we add a_j to S, rank of S increases. Then we need the following claim to be true.

CLAIM: a_j is not a linear combination os vectors in S.
PROOF: By contradiction assume there exists a way to express a_j as a linear combination $a_j = \sum_{i \in I} \lambda_i a_i$. We know that $a_j^T x \neq b_j$, because $j \notin I$. But also:

$$a_j^T (x + \lambda^* d) = b_j \Rightarrow a_j^T \neq b_j, a_j^T \lambda^* d \neq 0$$

,

but $a_j^T = 0 \forall i \in I$ by definition. So we know that $a_j^T \neq 0 and \sum_{i \in I} \lambda_i a_i^T d \Rightarrow 0 \neq a_j^T d = 0$. Contradiction.

Now get back to the proof of the theorem. The rank of S increases $rank(S) < rank(S \cup \{a_j\})$.

We take a point $x + \lambda^* d$ and move again. After we move n iterations $rank(S') = n$ we obtain the point x' with corresponding S" with $rank(S') = n$, hence we have n linearly independent constraints, hence x' is a vertex, hence also an extreme point.
$a) \Rightarrow c)$ Suppose x is an extreme point. That means that x is also basic feasible solution. Hence, there are n linearly independent constraints active at x.
$c) \Rightarrow b)$ Suppose we have n linearly independent constraints $a_1...a_n$. Suppose by contradiction p contains a line, means it contains all points of a form $\{x + \lambda d | \lambda \in \mathbb{R}, d \neq 0\}$. This also means that every point on this line satisfies all constraints:

$$a_i^T(x + \lambda d) \geq b_i \forall i, \lambda \in \mathbb{R}$$
$$a_i^T x + \lambda a_i^T d \geq b_i \forall i, \lambda \in \mathbb{R}$$

We want this to hold for all λ, so that it is not possible to pick arbitrary small negative λ, that would break the condition. Hence we need $\lambda a_i^T d = 0 \forall i$. But if $a_i^T d = 0 \forall i, d \neq 0$ then a_i are not linearly independent. Contradiction.

2.2 Optimal solution in extreme point

Theorem. Let $P \subset \mathbb{R}^n$ be polyhedron with at least one extreme point. Consider the LP $max\{c^T x | x \in P\}$ and assume a (finite) optimal solution exists. Then there exists optimal solution which is an extreme point.
Proof. Let $P = \{x \in \mathbb{R}^n | Ax \geq b\}$, v is a finite optimum $v = max\{c^T x | x \in P\}$. Define a new polyhedron Q, that contains only those points of P, that are optimal $Q = \{x \in \mathbb{R}^n | Ax \geq b \wedge c^T x = v\}$. P has an extreme point, thus it does not contain a line. Q is a subset of P, thus Q also does not contain a line. Hence Q has an extreme point x^*.

Claim: x^* is also an extreme point of P.
Proof: By contradiction assume that x^* is not an extreme point of P. then there exist two points of P, that give a linear combination of x^*: $\exists y, z \in P, \lambda \in [0, 1]$ such that $x^* = \lambda y + (1 - \lambda) z$. As $x^* \in Q$ we know $v = c^T x^* = \lambda c^T y + (1 - \lambda) c^T z$. As v is optimal solution then $c^T y \leq v, c^T z \leq v$.
If $y \in P \setminus Q$ then $c^T y < v \Rightarrow v = \lambda c^T y + (1 - \lambda) c^T z < v$. Then y gives strictly better solution for LP, hence $y \in Q \wedge z \in Q$. Hence x^* is not an extreme point of Q. Contradiction.

We know that x^* is an extreme point of P and $x^* \in Q$. Thus $c^T x^* = v$ and x^* is an optimal extreme point solution for LP.

3 Full rank assumption

Theorem. Let P $= \{x|Ax = b, x \geq 0\}$ where $A \in \mathbb{R}^{m\times n}$ but $rank(A) = k < m$. Assume $P \neq 0$ and w.l.o.g. that rows $a_1^T, ..., a_k^T$ are linearly independent. Define $Q = \{x|a_1^T x = b_1, ..., a_k^T x = b_k, x \geq 0\}$. Then Q = P.

Proof. 1) Every point that satisfies P also satisfies $Q \Rightarrow P \subseteq Q$.
2) Prove $Q \subseteq P$. Every row a_i^T of A can be expressed as $a_i^T = sum_{j=1}^k a_i^T \lambda_{ij}$ for some $\lambda \in \mathbb{R}$. Because $P \neq \varnothing$ we can say let $x \in P, b_i = a_i^T x = \sum_{j=1}^k a_i^T \lambda_{ij} = \sum_{j=1}^k \lambda_{ij} b_j \; \forall i$. Let $y \in Q \; \forall i$ $a_i^T y = \sum_{j=1}^k \lambda_{ij} a_i^T y = \sum_{j=1}^k \lambda_{ij} b_j = b_i$. Hence we know that y satisfied all constraints in Q. Then $y \in P$ and $Q \subseteq P$.
From 1) and 2) we conclude that $Q = P$.

From now on we consider that all A have full row rank.
Let $A \in \mathbb{R}^{m\times n}, m \leq n$. If x is a feasible solution, then first m constraints are tight, vector x is $m-$dimensional. $n - m$ constraints of a form $x_j \geq 0$ have to be also tight at x to satisfy n linear independence. How to choose these?

4 Extreme points of LP in standard form

Theorem. Given LP $Ax = b, x \geq 0$ and assume that rows of A are linearly independent. A vector $x \in \mathbb{R}^n$ is basic solution if and only if $Ax = b$ and there are indexes $B \subseteq \{1, ..., n\}, |B| = m$ such that:

a) the columns $A_j, j \in B$ are linearly independent

b) if $j \notin B$ then $x_j = 0$

Proof. 1) Direction $\Leftarrow$. Let $x^* \in \mathbb{R}^n$ such that $Ax = b$ and let B be a set of indexes satisfying a) and b). Consider a system of equations:

$$\begin{gathered} Ax = b \\ x_j = 0 \forall j \notin B \end{gathered}$$

We know that $b = Ax^* = \sum_{i=1}^n A_i x_i^*$. Let x be an arbitrary solution for the system. Then $b = Ax = \sum_{i=1}^n A_i x = \sum_{j\in B} A_j x_j$.
By assumption we know that A_j are linearly independent, hence the system has only one solution, that is x^*. Thus there are n linearly independent **tight** constraints at x^*. Then by definition x^* is a basic solution.
2) Direction $\Rightarrow$. Let x be a basic solution. Let's define a set of indexes B_1 such that $x_j \neq 0$. Consider the system of equations, that are tight at x. By assumption x is basic solution,

hence the system has a unique solution. Thus columns $A_j, j \in B$ are linearly independent. If not then there will be $\lambda_j \forall j \in B_1$ such that $\sum_{j \in B_1} A_j \lambda_j = 0$ where λ_j are not all 0. Then it would be $\sum_{j \in B_1} A_j(x_j + \lambda_j) = b$ and solution x is not unique in this case. Contradiction. Since row rank is the same as column rank $|B_1| \leq m$. And we know $rank(A) = m$, thus there exist m linearly independent columns. We can find $m - |B_1|$ columns B_2 with $B_1 \cap B_2 = 0$ such that columns represented by $B_1 \cup B_2$ are linearly independent $\Rightarrow$ a) is satisfied.
If there exist $j \notin B$ and $j \notin B_1$ then $x_j = 0$ by definition of B_1. b) is satisfied.