Optimization	**Summer 2016**
Lecture 8: Mai 18	
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1 Simplex Algorithm

1.1 Simplex Algotithm

x is basic solution for $min(c^T x)$ s.t. $Ax = b$ and $x \geq 0$ $(A \in \mathbb{R}^{n*n})$.

$$basis\ B = (B(1), ..., B(m))$$
$$columns\ A_{B(1)}, ..., A_{B(m)}\ are\ linearly\ independent.$$
$$if\ j \notin B(1), ..., B(m) - > x_j = 0(x_j\ is\ non\ basic\ variable)$$
$$Ax = b$$
$$x_B = (X_{B(1)}, ..., X_{B(n)})(those\ are\ the\ basic\ variables)$$
$$A_B = (A_{B(1)}, ..., A_{B(m)})$$
$$X_B = A_B^{-1} * b$$

From point x we now want go to $x + \theta d$ with $\theta \geq 0$. For each $j \in [n] \setminus B$ we define d^j

$$d_j^j = 1$$
$$d_i^j = 0\ (for\ each\ i \in [n](B \cup \{j\})$$
$$d_B^j = -A_B^{-n} * A_j$$

It follows

$$Ad^j = 0$$
$$Ax = b$$
$$A(x + \theta d) = Ax + \theta Ad = b\ as\ \theta Ad = 0$$

we now define the reduced cost $\bar{c_j}$ as

$$c^T(x + \theta d^j) - c^T x = c_j - c_B^T A_B^{-1} A_j = \bar{c_j}$$

if the result vector $\bar{c}$ is strictly positiv $(\bar{c} \geq 0)$ then x is the optimal solution.

The resulting Simplex Algorithm is as follows:

1. start with basic feasable solution x and the corresponding basis B
2. compute $\bar{c}$
3. if $\bar{c} \geq 0$
   - TRUE then x is optimal $\Rightarrow$ STOP
   - FALSE assume $\bar{c_j} < 0$
     move from x in direction d^j by moving to point $x + \theta^* d^j$ where
     $\theta^* = max\{\theta \geq 0 | x + \theta d^j \in P\}$ non negativity constraints are satisfied at $x + \theta d^j$
     for the non-negativity constraints:
     (a) $d^j \geq 0$ then $x + \theta d^j \geq 0$ for any $\theta \geq 0 c^T(x + \theta d^j) = c^T x + \theta c^T d^j$.
     It follows that optimal solution is $-\infty$. STOP
     (b) if $d_i^j < 0$ for some $i, x_i + \theta d_i^j \geq 0$

$$
\begin{array}{l}
\theta^* = min_{\{i|d_i^j<0\}} - \frac{-x_i}{d_i^j} \iff 0 \leq \frac{-x_i}{d_i^j} \\
\theta > 0 : d_i^j \geq 0 \; if \; x_i \; is \; nonbasic \; variable \\
x_i > 0 \; if \; i \in B(x \; is \; non-degenerate) \\
\\
let \; B(l) \; be \; minimize \; for \; (*) \; it \; follows : \\
y = x + \theta^* d^j \\
\theta^* = \frac{-x_{B(l)}}{d_{B(l)}^j} \\
y_{B(l)} = X_{B(l)} + \theta^* d_{B(l)}^j = 0 \; and \; d_{B(l)}^j < 0 \\
new \; basis \; \bar{B} : \bar{B}(i) = \left\{ \begin{array}{c} B(i) if i \neq l \\ j if i = l \end{array} \right. \quad \theta^* > 0 \\
\Rightarrow y \neq x \\
\Rightarrow c^T y < c^T x
\end{array}
$$

1.2 Theorem 42

1. the columns of $A_{\bar{B}}$ are linearly independent
2. y is basic feasable solution corresponding to $\bar{B}$

Proof of (a)

$$
\begin{array}{l}
assume \; that \; A_{B(1)}, ..., A_{B(m)} \; are \; linearly \; independent \\
\Rightarrow \exists \; coefficient \; \lambda_1, ..., \lambda_m \; with \; not \; all \lambda = 0 \\
\\
\sum_{i=1}^m \lambda_i A_{\bar{B}(i)} = 0 \\
\Rightarrow \sum_{i=1}^m \lambda_i A_B^{-1} A_{\bar{B}(i)} = 0 \\
vectors \; A_B^{-1} A_{B(i)}, ..., A_B^{-1} A_{B(m)} are \; linearly \; dependent
\end{array}
$$

$A_B^{-1} * A_B = I = A_B^{-1} * A_{B(i)} = l_i$
$A_{B(i)} = A_{\bar{B}(i)}\ for\ all\ i \neq l$
$\Rightarrow A_B^{-1} * A_{\bar{B}(i)} = A_B^{-1} * A_{B(i)} = e_i\ if\ i \neq l$
$\bar{B}(l) = j$
$A_B^{-1} A_{\bar{B}(l)} = A_B^{-1} A_j = -d_B^j (the\ l^{th}\ entry\ of\ d_B^j = d_{B(l)}^j$

$$A_B^{-1} A_B = \begin{pmatrix} 1 \ \ldots\ldots ? \ldots\ldots\ldots \\ \ldots \ddots \ldots ? \ldots\ldots\ldots \\ \ldots\ldots\ 1\ ? \ldots\ldots\ldots \\ \ldots\ldots\ldots ?\ 1\ \ldots\ldots \\ \ldots\ldots\ldots ? \ldots \ddots \ldots \\ \ldots\ldots\ldots ? \ldots\ldots\ 1 \end{pmatrix}$$

$det(A_B^{-1} A_{\bar{B}}) \neq 0$
$\Rightarrow\ vectors\ A_B^{-1} A_{B(i)}, ..., A_B^{-1} A_{B(m)}\ are\ linearly\ independent.\ Contradiction!$

Proof of (b)

$y \geq 0, Ay = b, y_i = 0\ if\ i \notin \bar{B}$
$columns\ of\ A_{\bar{B}}\ are\ linearly\ independent$
$\Rightarrow y\ is\ basic\ feasable\ solution\ for\ basis\ \bar{B}$

1.3 One Iteration of Simplex

One iteration of Simplex algorithm is called a pivot.

1. start with basis matrix A_B, defining basic feasable solution x
2. compute reduced costs $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$
   (a) if $\bar{c}_j \geq 0 \Rightarrow x$ optimal solution. STOP
   (b) choose some j with $\bar{c}_j < 0$
3. compute $u = A_B^{-1} A_j = -d_B^j$ if $u \leq 0$ the optimum is $-\infty$. STOP
4. choosse index l such that $\frac{X_{B(l)}}{u_l} = \theta^* = min\{\frac{X_{B(i)}}{u_i} | i \in [n] and u_i > 0\}$
5. Form new basis $\bar{B}$ by replacing $B(l)$ with j
6. new basic feasable solution y with

   $y_i = \theta^*$

$y_i = 0\ if\ i \notin \bar{B} = B \cup \{j\} \setminus B(l)$
$y_{B(i)} = X_{B(i)} - \theta^* * u_i$

1.4 Theorem 43

Assume $P \neq \emptyset$, every basic feasable solution is non-degenerate, and that the algorithm is initialized with a basic feasable solution. Then it terminates after a finite amount of iterations. At termination, there are the following two options:

- We have a optimal Basis B and a associated basic feasible solution that is optimal
- We have a vector d satisfying $Ad = 0, d \geq 0, c^T d < 0$ and thus the optimal cost is $-\infty$.

Proof

- If algorithm terminates in step 2 the solution is optimal because $\bar{c} \geq 0$
- If algorithm terminates in step 3
  $\Rightarrow \exists$ basic feasible solution x and direction d^j with $Ad^j = 0, x + \theta d^j \in P, \forall \theta \geq 0$
  $c^T d^j = \bar{c}_j < 0$
  cost of $x + \theta d^j$ is $c^T(x + \theta d^j) = c^T x + \theta c^T d^j \Rightarrow\ The\ optimum\ is - \infty$.

in each pivot the objective value strictly decreases. Thus, no vertice is visited twices. As there are only a limited amount of vertices the algorithm has to terminate after a finite amount of iterations.