Optimization	**Summer 2016**

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1 Simplex Algorithm

1.1 Iteration of Simplex Algorithm (a "pivot")

1. start with basis Matrix $A_{B(1)}$,...,$A_{B(m)}$ $\rightarrow$ basic feasible solution x.
2. compute reduced costs $\bar{c}_j = c_j - c_B^T A_B^{-1} A_j$ for each nonbasic variable x_j.
   (a) if all $\bar{c}_j \geq 0$ then we are optimal.
   (b) choose some j with $\bar{c}_j < 0$
3. compute $u = A_B^{-1} A_j = -d_B^j$. If $u \leq 0$ then the optimum is $-\infty$ and we stop.
4. Choose index l such that $u_l > 0$ and

   $\frac{x_{B(l)}}{u_l} = \theta^* = min\{\frac{x_{B(i)}}{u_i} | i \in [m] \text{ and } u_i > 0\}$
5. Form new basis by replacing $A_{B(l)}$ with A_j.

1.2 Faster Implementation

1st iteration: basic feasible solution x, basis matrix A_B, compute A_B^{-1}.
2nd iteration: basic feasible solution $\bar{x}$, basis matrix $A_{\bar{B}}$, compute $A_{\bar{B}}^{-1}$
$\rightarrow$ derive $A_{\bar{B}}^{-1}$ from A_B^{-1}.

We know, that A_B and $A_{\bar{B}}$ are very similar.
Idea: Are A_B^{-1} and $A_{\bar{B}}^{-1}$ also similar?

We also know: $A_B^{-1}A_B = I$, $A_B^{-1}A_{B(i)} = e_i$, $A_B^{-1}A_j = u$

$$A_B^{-1}A_{\bar{B}} = \begin{bmatrix} 1 & 0 & \dots & u_1 & \dots & 0 & 0 \\ 0 & \ddots & \dots & u_2 & \dots & 0 & 0 \\ \vdots & \vdots & 1 & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & u_l & 0 & \dots & 0 \\ \vdots & & & \vdots & 1 & & \vdots \\ 0 & & & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & u_m & \dots & 0 & 1 \end{bmatrix}$$

We want to find a matrix Q, such that $\underbrace{QA_B^{-1}}_{=A_{\bar{B}}^{-1}} A_{\bar{B}} = I$.

1.3 Elementary Row Operations

Multiply i-th row by some $\alpha \neq 0$
$\Leftrightarrow$ multiplying from left with

$$Q_1 = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & & \vdots \\ \vdots & \ddots & 1 & 0 & & & \vdots \\ \vdots & & 0 & \alpha & 0 & & 0 \\ \vdots & & & 0 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix}$$

This matrix is like the unit matrix, but with α at position (l, l). Obviously Q_1 is invertible.

Now add β times the j-th row to the i-th row for $i \neq j$ to eliminate the non-diagonal components of u:

$$Q_2 = \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & & \beta & \vdots \\ \vdots & \ddots & 1 & 0 & & & \vdots \\ \vdots & & 0 & 1 & 0 & & 0 \\ \vdots & & & 0 & 1 & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix}$$

This matrix is like the unit matrix, but with β at position (i, j).

With these elementary row operations turn $A_B^{-1} A_{\bar{B}}$ into I.

- For each $i \neq l$:
  - Add the l-th row $-\frac{u_i}{u_l}$ times to the i-th row.
- Multiply l-th row by $\frac{1}{u_l}$

In other words: find $Q_1, \dots, Q_m$ such that:

$$\underbrace{Q_m \dots Q_2 Q_1}_{Q} A_B^{-1} A_{\bar{B}} = I \implies Q A_B^{-1} = A_{\bar{B}}^{-1}$$

1.4 Simplex: full tableau implementation

$-c_B^T x_B$	$\bar{c_1}$	$\dots$	$\bar{c_n}$
$x_{B(1)}$			
$\vdots$	$A_B^{-1} A_1$	$\dots$	$A_B^{-1} A_n$
$x_{B(m)}$			

Note that $-c_B^T x_B$ is the negated objective value of x, $\bar{c_1}, \dots, \bar{c_n}$ the reduced costs and the vectors $A_B^{-1} A_l = u_n$.
We call the vector $(-c_B^T x_B, \bar{c_1}, \dots, \bar{c_n})^T$ the 0-th row of our tableau and the vector $(-c_B^T x_B, x_{B(1)}, \dots, x_{B(m)})^T$ the 0-th column.

1.5 Pivot step

1. If $\bar{c} \geq 0$ then STOP.
   Otherwise choose j such that $\bar{c_j} \leq 0$.

2. consider $u = A_B^{-1} A_j$.
   If $u \le 0$ then STOP.

3. for each i with $u_i > 0$ compute $\frac{x_{B(i)}}{u_i}$
   Let l be the index of a row that minimizes this ratio.

4. Column A_j enters basis,
   column $A_{B(l)}$ leaves basis.

5. perform elementary row operation such that:

   (a) u_l becomes 1

   (b) all other entries in the j-th column become 0, including entries in 0-th row.

1.5.1 Example

$$
\begin{array}{lccccccccccccc}
\text{minimize} & -10x_1 & - & 12x_2 & - & 12x_3 & & & & & & & & \\
\text{subject to} & x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & & & & & = & 20 \\
& 2x_1 & + & x_2 & + & 2x_3 & & & + & x_5 & & & = & 20 \\
& 2x_1 & + & 2x_2 & + & x_3 & & & & & + & x_6 & = & 20
\end{array}
$$

Initial solution $x = (0, 0, 0, 20, 20, 20)$
$B(1) = 4$, $B(2) = 5$, $B(3) = 6$
$A_B = I = A_B^{-1}$, $c_B = 0$

The tableau for this LP looks like this:

0	−10	−12	−12	0	0	0
20	1	2	2	1	0	0
20	2*	1	2	0	1	0
20	2	2	1	0	0	1

$*$ this is the u_l that has to become 1. The other entries in this column ($= u$) have to become 0.

$\frac{x_{B(1)}}{u_1} = \frac{x_4}{u_1} = 20$

$\frac{x_{B(2)}}{u_2} = \frac{x_5}{u_2} = 10$

$\frac{x_{B(3)}}{u_3} = \frac{x_6}{u_3} = 10$

Index $l = 2 \implies A_1$ enters the basis, A_5 leaves the basis
$B(1) = 4$, $B(2) = 1$, $B(3) = 6$

100	0	-7	-2	0	5	0
10	0	$0,5$	1	1	$-0,5$	0
10	1	$0,5$	1	0	$0,5$	0
0	0	1	-1	0	-1	1

1.6 Lemma 44

The elementary row operations lead to tableau

$-c_{\bar{B}}^T A_{\bar{B}}^{-1} b$	$c^T - c_{\bar{B}}^T A_{\bar{B}}^{-1} A$
$A_{\bar{B}}^{-1} b$	$A_{\bar{B}}^{-1} A$

where $\bar{B}$ is obtained from adding j to B and removing $B(l)$ from B

1.6.1 Proof

Entries $A_{\bar{B}}^{-1} b$ and $A_{\bar{B}}^{-1} A$:
Elementary row operations are equivalent to left-multiplying with matrix Q such that $QA_B^{-1} = A_{\bar{B}}^{-1}$.
0-th row:
we started with $[0|c^T] - \underbrace{g^T[b|A]}_{\substack{\text{linear}\\ \text{combination}\\ \text{of rows of A}}}$
with $g^T = c_B^T A_B^{-1}$.
After iteration, 0-th row equals $[0|c^T] - p^T[b|A]$.
$\implies c_j - p^T A_j = 0$ where $j = \bar{B}(l)$
Let $i \neq l$, $\bar{c}_{B(i)} = 0$ and entry stays 0 after update.
$c_{\bar{B}}^T - p^T A_{\bar{B}} = 0 \implies p^T = c_{\bar{B}}^T A_{\bar{B}}^{-1}$
$\implies$ 0-th row equals $[0|c^T] - c_{\bar{B}}^T A_{\bar{B}}^{-1} [b|A]$ $\square$