Optimization	**Summer 2016**

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1 One iteration of the Simplex algorithm a.k.a. a "pivot"

1.1 Pseudocode

1. Start with basis matrix $A_{B(1)}, \ldots, A_{B(m)} \rightarrow$ basic feasible solution $\mathbf{x}$.
2. Compute reduced costs $\bar{c}_j = c_j - c_j^{\intercal} A_B^{-1} A_j$ for each nonbasic variable x_j.
   (a) if all $\bar{c}_j \geq 0$ then we are optimal. STOP.
   (b) choose *some* j with $\bar{c}_j < 0$
3. Compute $u = A_B^{-1} A_j = -d_B^j$. If $u \leq 0$ then optimum $= -\infty$ and we stop.
4. Choose index l such that $u_l > 0$ and

$$\frac{x_{B(l)}}{u_l} = \theta^* = min \left\{ \frac{x_{B(i)}}{u_i} \mid i \in [m] \text{ and } u_i > 0 \right\} \tag{1}$$

5. Form new basis by replacing $A_{B(l)}$ with A_j.

1.2 Introduction

The computational most expensive operation used in a pivot is the generation of A_B^{-1} which has to be recomputed whenever the basis B changes. Computing the inverse of a matrix costs $O(n^3)$ by applying brute-force, e.g. Gauss-Elimination.

We observe that only one column of B will change per pivot. Assume that A_B^{-1} is at our disposal from the last pivot and we want to compute $A_{\bar{B}}^{-1}$ where $\bar{B}$ denotes a new basis.

The idea we get is that $A_{\bar{B}}^{-1}$ might look similar to A_B^{-1} if $A_{\bar{B}}$ and A_B^{-1} differ only by one column. This lecture introduces a method exploiting elementary row operations within a *tableau*-scheme to compute $A_{\bar{B}}^{-1}$ from A_B^{-1} in $O(n^2)$.

1.2.1 Mathematical point of view

$$
\begin{aligned}
A_B^{-1} A_B &= I \\
A_B^{-1} A_{B(i)} &= e_i \\
A_B^{-1} A_j &= u
\end{aligned}
$$

$$
A_B^{-1} A_{\bar{B}} = \begin{bmatrix}
1 & 0 & 0 & \cdots & u_1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & u_2 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & u_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & u_l & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & u_{n-1} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & u_n & 0 & 0 & 1
\end{bmatrix} \tag{2}
$$

Our goal is to find a matrix Q such that $QA_B^{-1}A_{\bar{B}} = I$. In other words we search for elemantary row operations transforming A_B^{-1} into $A_{\bar{B}}^{-1}$ via Q ($QA_B^{-1} = A_{\bar{B}}^{-1}$).

1.3 Elementary row operations

We will use elementary row operations to setup the matrix Q. A short recap, there are three different elementary row operations:

1. Switching: A row within the matrix can be switched with another row,
2. Multiplication: Components of a row-vector may be multiplicated with a non-zero scalar $\lambda \in \mathbb{R}$,
3. Addition: Multiples of a row j can be added to another row i where $i \neq j$.

Starting from equation 2 our first goal is to bring the entry $(l, j) = u_l$ to 1. Multiplying i-th row by some $\alpha \neq 0$ is equivalent to multiply with Q_1 from left. For simplicity we assume $l = j$ here.

$$
Q_1 = \begin{array}{cc}
\begin{array}{cccccccc} & & & & l & & & \end{array} & \\
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} & \begin{array}{c} \\ \\ \\ l \\ \\ \\ \end{array}
\end{array} \tag{3}
$$

Since Q_1 is a square matrix with full rank, it is invertible. Another argumentation is that $det(Q_1) \neq 0$.

To eliminate the non-diagonal entries of u a multiple of the l-th row can be subtracted from the i-th row. Written in terms of a matrix operation multiplied from left:

$$Q_2 = \begin{array}{cc} \begin{array}{cccccc} & & & & j & \end{array} & \\ \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \beta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \\ i \\ \\ \\ \\ \\ \end{array} \end{array} \tag{4}$$

In summary, elementary row operations will help us to turn $A_B^{-1}A_{\bar{B}}$ into I.

- For each $i \neq l$ add l-th row $-\frac{u_i}{u_l}$ times to the i-th row.
- Multiply l-th row by $\frac{1}{u_l}$

In matrix notation we need a series of matrices Q_i such that:

$$\underbrace{Q_m \cdots Q_2 Q_1}_{Q} A_B^{-1} A_{\bar{B}} = I \implies QA_B^{-1} = A_{\bar{B}}^{-1} \tag{5}$$

Implementation of the elementary row operations to accelerate the Simplex method is often done in a tableau-scheme.

1.4 Simplex: A full tableau implementation

Idea: Every iteration of the Simplex algorithm maintains a tableau (matrix) which stores important properties of the current Simplex iteration.

$$\begin{array}{cc|ccc} & \overbrace{\hspace{4em}}^{\substack{\text{Negated} \\ \text{objective} \\ \text{value of } \mathbf{x}}} & \multicolumn{3}{c}{\overbrace{\hspace{14em}}^{\text{Reduced costs}}} \\ & -c_B^\intercal \mathbf{x}_B & \bar{c}_1 & \cdots & \bar{c}_n \\ \hline x_B & \left\{ \begin{array}{c} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{array} \right. & \underbrace{A_B^{-1} A_1}_{u_1} & \cdots & \underbrace{A_B^{-1} A_n}_{u_n} \end{array}$$

1.5 Pivot step

1. If $\bar{c} \geq 0$ then STOP (costs already computed by previous iteration). Otherwise choose j such that $\bar{c}_j < 0$.

2. Consider $u = A_B^{-1} A_j$ (is already in tableau from previous step). if $u \leq 0$ then STOP (Move to new vertex or equivalently switch basis).

3. For each i with $u_i > 0$ compute $\frac{x_{B(i)}}{u_i}$. Let l be the index of a row minimizing this ratio.

4. Column A_j enters the basis and column $A_{B(l)}$ leaves the basis (We have to update the tableau now by performing elementary row operations).

5. Performing elementary row operations such that: Turn u_l into u_i by transforming u_l to 1 and all other entries $i \neq l$ of the j-th column to 0.

1.5.1 Example

$$
\begin{array}{llllllllllllll}
\text{min} & -10 & x_1 & -12 & x_2 & -12 & x_3 & & & & & & & \\
\text{subject to} & 1 & x_1 & +2 & x_2 & +2 & x_3 & +1 & x_4 & & & & = & 20 \\
 & 2 & x_1 & +1 & x_2 & +2 & x_3 & & & +1 \; x_5 & & & = & 20 \\
 & 2 & x_1 & +2 & x_2 & +1 & x_3 & & & & +1 & x_6 & = & 20 \\
 & & x_1 & , & x_2 & , & x_3 & , & x_4 & , \; x_5 & , & x_6 & \geq & 0
\end{array}
$$

Table 1: LP for the tableau method example.

0	-10	-12	-12	0	0	0
20	1	2	2	1	0	0
20	2	1	2	0	1	0
20	2	2	1	0	0	1

Table 2: Initialize tableau.

Initial solution $\mathbf{x} = (\underbrace{0,0,0}_{\text{Non-basic variables}}, \underbrace{20,20,20}_{\text{Basic variables}})$, $B(1) = 4$, $B(2) = 5$, $B(3) = 6$. $A_B = I = A_B^{-1}$, $c_B = 0$. Perform 3rd pivot step (see equation 1 or enumeration 1.5).

$$
\begin{aligned}
\frac{x_{B(1)}}{u_1} &= \frac{x_4}{u_1} &= 20 \\
\frac{x_{B(2)}}{u_2} &= \frac{x_5}{u_2} &= 10 \\
\frac{x_{B(3)}}{u_3} &= \frac{x_6}{u_2} &= 10
\end{aligned}
\tag{6}
$$

Minimal index $l = 2 \implies A_1$ enters the basis and A_5 leaves the basis. New basis: $B(1) = 4$,

100	0	-7	-2	0	5	0
10	0	0.5	1	1	-0.5	0
10	1	0.5	1	0	0.5	0
0	0	1	-1	0	-1	1

Table 3: Tableau after 1st iteration.

$B(2) = 1$, $B(3) = 6$.

1.6 Lemma 44

The elementary row operations lead to tableau: where $\bar{B}$ is obtained from adding j to B

$-c_{\bar{B}}^\intercal A_{\bar{B}}^{-1} b$	$c^\intercal - c_{\bar{B}}^\intercal A_{\bar{B}}^{-1} A$
$A_{\bar{B}}^{-1} b$	$A_{\bar{B}}^{-1} A$

Table 4: Tableau after performing elementary row operations.

and removing $B(l)$ from B.

1.6.1 Proof

Consider entries of $A_{\bar{B}}^{-1}$ and $A_{\bar{B}}^{-1} A$:
Elementary row operations are equivalent to left-multiplying with a matrix Q such that $QA_B^{-1} = A_{\bar{B}}^{-1}$.

0-th row: We started with $[\underbrace{0}_{\substack{\text{First}\\\text{entry}}} \mid \underbrace{c^\intercal}_{\substack{\text{Second}\\\text{entry}}}] - \underbrace{g^\intercal [b \mid A]}_{\substack{\text{Linear}\\\text{combination}\\\text{of rows of } A.}}$. with $g^\intercal = c_B^\intercal A_B^{-1}$.

After the i-th iteration: $[0 \mid c^\intercal] - p^\intercal [b \mid A] \implies c_j - p^\intercal A_j = 0$ where $j = \bar{B}(l)$. Consider columns of old basis $\to$ identity matrix. l-th row was zero at column $j \to$ new reduced cost stays 0. Let $i \neq l : \bar{c}_{B(i)} = 0$ and entry stays 0 after update. $c_{\bar{B}}^\intercal - p^\intercal A_{\bar{B}} = 0 \implies p^\intercal = c_{\bar{B}}^\intercal A_{\bar{B}}^{-1} \implies 0$ $\square$