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Complexity Theory of Polynomial-Time Problems

Lecture 7: 3SUM II

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Reminder: 3SUM

given sets A, B, C of n integers

are there $a \in A, b \in B, c \in C$ such that $a + b + c = 0$?

well-known: $O(n^2)$

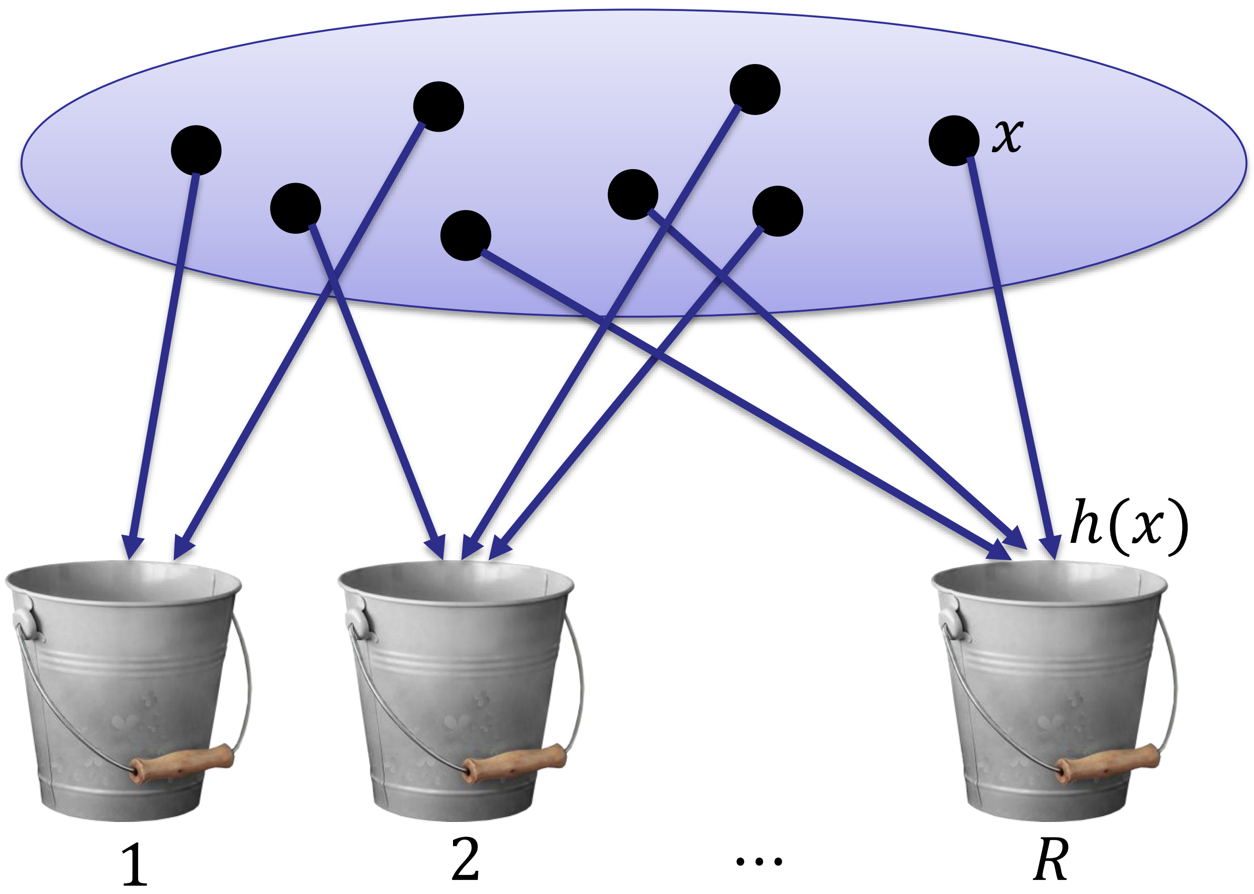
Conjecture: no $O(n^{2-\varepsilon})$ algorithm

→ 3SUM-Hardness

Alternative algorithm: $O(|A| \cdot |B| + |C|)$
(store negated pairwise sums in hashmap)

Reminder: Hashing

Hash function $h: [U] \to [R]$



Goal: Distribute uniformly, avoid collisions, etc.

Magical hash functions

Desired properties for family of hash functions from $[U] \to [R]$
(i.e., for every h chosen from family)

Uniform difference: $\Pr[h(x) - h(y) = i] = 1/R$

(for any $x, y \in [U]$ s.t. $x \neq y$ and $i \in [R]$)

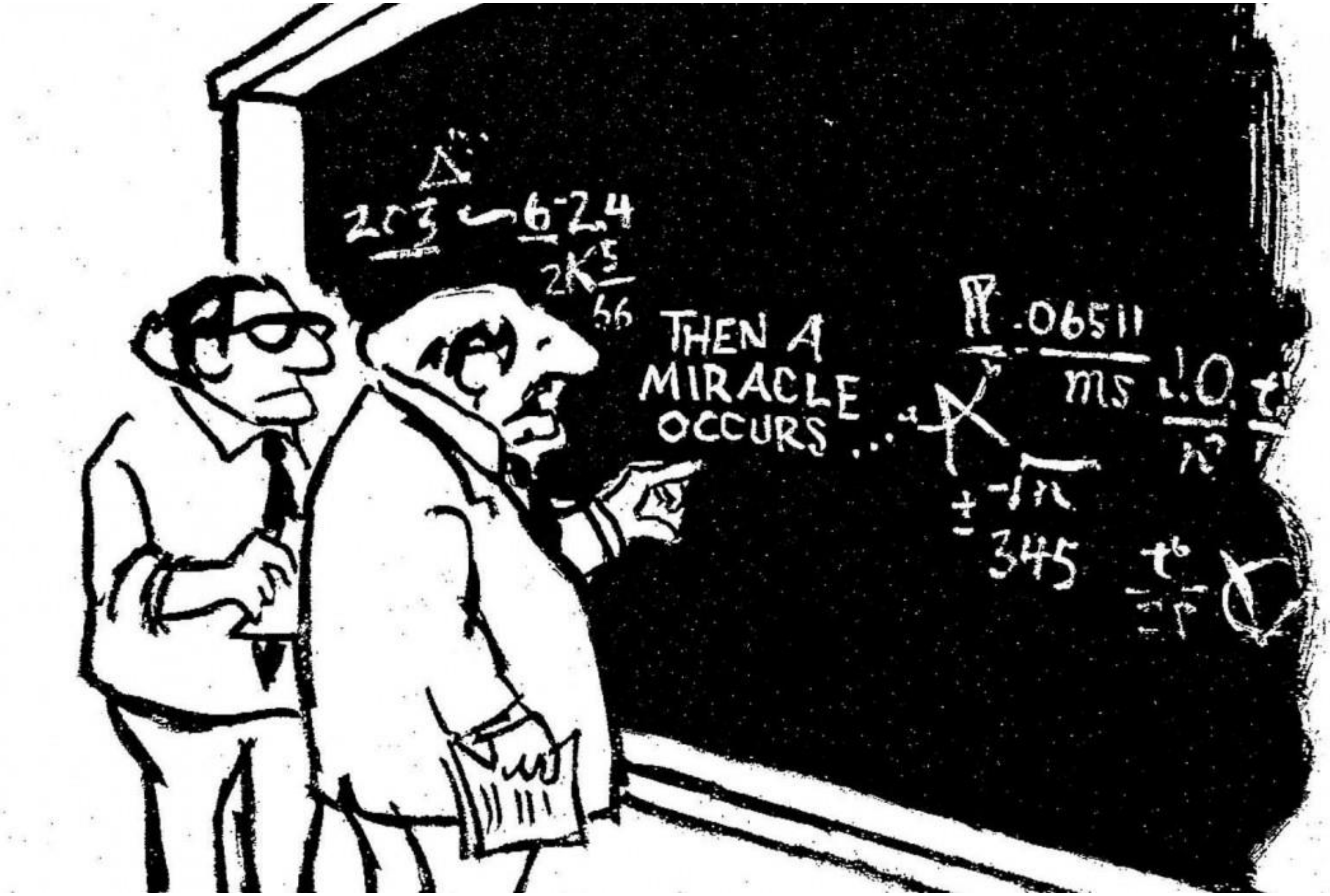
Balanced: $|\{x \in S : h(x) = i\}| \leq 3n/R$

(for any set $S = \{x_1, \ldots, x_n\} \subseteq [U]$ and any $i \in [R]$)

Linear: $h(x) + h(y) = h(x + y) \pmod R$

(for any $x, y \in [U]$)

But: We do not know such a family…


THEN A MIRACLE OCCURS . . .

Almost magical hash functions

Desired properties for family of hash functions from $[U] \to [R]$
(i.e., for every h chosen from family)

Uniform difference: $\Pr[h(x) - h(y) = i] = 1/R$

(for any $x, y \in [U]$ s.t. $x \neq y$ and $i \in [R]$)

Almost balanced: Expected number of elements from S hashed to heavy values is $O(R)$, where value $i \in [R]$ is heavy if $|\{x \in S : h(x) = i\}| > 3n/R$

(for any set $S = \{x_1, \dots, x_n\} \subseteq [U]$ and any $i \in [R]$)

Almost linear: $h(x) + h(y) \in h(x + y) + c_h + \{0,1\} \pmod{R}$

(for any $x, y \in [U]$ and some integer c_h depending only on h)

Definition of hash function

Set $r = km$ for some $k \geq U/2$ and U, R, r powers of 2

$$\mathcal{H}_{U,R,r} = \{h_{a,b}: [U] \rightarrow [R] \mid a \in [r] \text{ odd integer and } b \in [r]\}$$

$$h_{a,b}(x) = (ax + b \bmod r) \operatorname{div} (r/R)$$

Thm: Family $\mathcal{H}_{U,R,r}$ is has the uniform difference property, is almost balanced and almost linear with $c_{h_{a,b}} = (b - 1 \bmod r) \operatorname{div} (r/R)$.

(Pairwise independence [Dietzfelbinger '96] implies uniform difference (easy to check) and almost balanced [Baran et al. '08]. Almost linear: easy to check.)

Rest of this lecture: h picked randomly from this family

Hashing down the universe

Lem: If 3SUM on universe of size $O(n^3)$ solvable in exp. time $O(n^{2-\epsilon})$, then 3SUM on arbitrary universe solvable in expect. time $O(n^{2-\epsilon})$.

Follows from [Baran et al. '08]

Algorithm:

Repeat until output:

- Pick hash function $h: [1 \ldots U] \rightarrow [1 \ldots 6n^3]$ at random
- $A' = \{ h(a) \mid a \in A \}$, $B' = \{h(b) \mid b \in B\}$, $C' = \{ h(c) + c_h \mid c \in C \}$
- $A'' = \{ h(a) \mid a \in A \}$, $B'' = \{h(b) \mid b \in B\}$, $C'' = \{ h(c) + c_h + 1 \mid c \in C \}$
- Solve two 3SUM instances (A', B', C') and (A'', B'', C'')
- If algorithm reports no 3SUM witness: output 'no 3SUM'
- Consider first reported 3SUM witness x', y', z' for (A', B', C'):
  - If $h^{-1}(x'), h^{-1}(y'), h^{-1}(z' - c_h)$ contains witness x, y, z: output x, y, z
- Consider first reported 3SUM witness x'', y'', z'' for (A'', B'', C''):
  - If $h^{-1}(x''), h^{-1}(y''), h^{-1}(z'' - c_h - 1)$ contains witness x, y, z: output x, y, z

No false negatives: If $x + y = z$, then $h(x) + h(y) \in h(z) + c_h + \{0,1\}$

Running Time

We need to bound:

- Number of iterations $O(1)$
- Number of candidate witnesses $O(1)$

Then: number of calls to 3SUM algorithm: $O(1)$

Number of iterations:

Triple x, y, z gives false positive if $x + y \neq z$ and one of

$$h(x) + h(y) = h(z) + c_h \text{ or } h(x) + h(y) = h(z) + c_h + 1$$

Linearity: $h(x) + h(y) = h(x + y) + c_h$ or $h(x) + h(y) = h(x + y) + c_h + 1$

Thus, probability that fixed x, y, z (with $x + y \neq z$) gives **false positive** is:

$$\Pr[h(x + y) - h(z) \in \{-1,0,1\}] \leq \frac{3}{6n^3} = \frac{1}{2n^3}$$ *(uniform difference)*

Overall probability of **false positive**: $\leq n^3 \cdot \frac{1}{2n^3} = \frac{1}{2}$

In expectation: 2 iterations until no false positive *(waiting time bound)*

(If no false positive, then algorithm certainly stops)

Running Time

We need to bound:
- Number of iterations $O(1)$
- Number of candidate witnesses $O(1)$

Then: number of calls to 3SUM algorithm: $O(1)$

Number of candidate witnesses:

Fix 3SUM witness x', y', z' of instance (A', B', C')

Let $x^* \in h^{-1}(x')$

For every $x \neq x^*$: $\Pr[h(x) = h(x^*)] = \frac{1}{6n^3}$ *(uniform difference)*

$E[|h^{-1}(x')|] \leq 1 + \frac{n}{4n^3} \leq 2$

Similarly: $E[|h^{-1}(y')|] \leq 2$, $E[|h^{-1}(z')|] \leq 2$

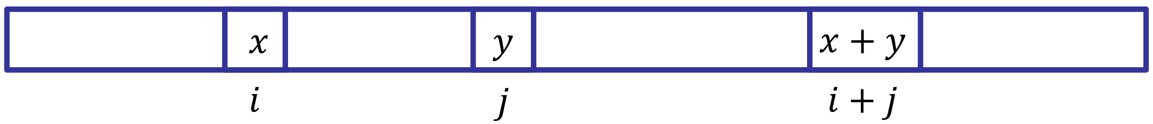
$E[|h^{-1}(x') \cup h^{-1}(y') \cup h^{-1}(z')|] \leq O(1)$ *(linearity of expectation)*

In expectation, algorithm manually checks constant number of candidate witnesses per iteration

Convolution 3SUM

Given array $A[1 \dots n]$ of integers

are there i, j such that $A[i] + A[j] = A[i + j]$?



trivial algorithm: $O(n^2)$

Thm: There is no $O(n^{2-\epsilon})$ algorithm for Convolution 3SUM unless the 3SUM Conjecture fails.

[Pătrașcu 2010]

Stepping stone towards hardness of other “structured” problems

Reduction from 3SUM

Given set $X \subseteq [U]$ of integers

are there $x, y, z \in X$ such that $x + y = z$?

Preprocessing: Check if there is a solution $2x = z$ $O(n \log n)$

Pick random hash function $h: [U] \to [R]$ (almost linear, etc.)

For this proof: assume h is almost balanced and linear *(magically…)*

$3n/R$

$\vdots$

1

1 2 $\cdots$ R

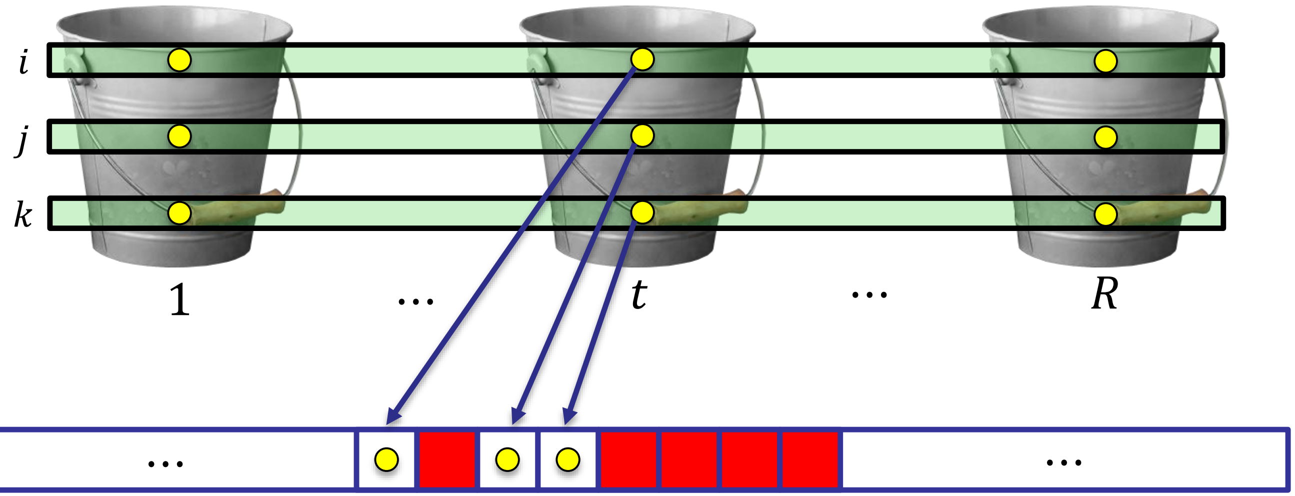
In expectation: $O(R)$ elements in buckets with load $> 3n/R$ *(almost bal.)*

For each such x: check for 3SUM triple involving x $O(Rn)$ (in exp.)

Convolution 3SUM instance

Number elements in each bucket from 0 to $\frac{3n}{R} - 1$

Iterate over all triples $i, j, k \in [3n/R]$



For every bucket t:

- Put i-th element to $A[8t + 1]$
- Put j-th element to $A[8t + 3]$
- Put k-th element to $A[8t + 4]$

Set all other array entries to ∞ (sufficiently large number)

$\left(\frac{3n}{R}\right)^3$ instances of Convolution 3SUM

Correctness

Assume $x + y = z$

Then $h(x) + h(y) = h(z) \pmod{R}$ *(linearity)*

If $x = y$, triple found in preprocessing

If x, y, or z hashed to heavy bucket: triple found in second step

Either $h(x) + h(y) = h(z)$ or $h(x) + h(y) = h(z) + R$

Duplicate array for Convolution 3SUM instance

$A[8h(x) + 1] + A[8h(y) + 3] = A[8h(z) + 4]$ or
$A[8h(x) + 1] + A[8h(y) + 3] = A[8(h(z) + R) + 4]$

Thus, no false negatives. Also no false positives:

Observation: $A[i] + A[j] = A[i + j]$ only if $i = 8t_1 + 1$ and $j = 8t_2 + 3$

($x + y = z \pmod{8}$ has unique solution over $\{1,3,4\}$ and $A[i] \neq A[j]$)

Running Time

Assumption: Convolution 3SUM in time $O(n^{2-\epsilon})$

Total expected running time: $O\left(n \log n + nR + \left(\frac{n}{R}\right)^3 n^{2-\epsilon}\right)$

Set $R = n^{1-\epsilon/4}$

Total time: $O\left(n^{2-\epsilon/4}\right)$

Contradicts 3SUM Conjecture

Set Disjointness Problem

1. **Preprocess** subsets $\mathcal{A}, \mathcal{B} \subseteq U$ over universe U

2. Answer **queries**: Given $A \in \mathcal{A}, B \in \mathcal{B}$, is $A \cap B \neq \emptyset$?

Repeated queries

(Static) data structure

Queries not known in advance

Goal: Lower bound on preprocessing **and** query time

Offline Set Disjointness: q queries known in advance (part of input)

Reduction to 3SUM [Kopelowitz et al]

Thm: Let $f(n)$ be such that 3SUM requires expected time $\Omega(n^2/f(n))$. For any constant $0 \leq \gamma < 1$, let ALG be an algorithm for offline Set Disjointness where $|\mathcal{A}| = |\mathcal{B}| = \Theta(n \log n)$, $|U| = \Theta(n^{2-2\gamma})$, each set in $\mathcal{A} \cup \mathcal{B}$ has at most $O(n^{1-\gamma})$ elements from U, and $q = \Theta(n^{1+\gamma} \log n)$. Then ALG requires expected time $\Omega(n^2/f(n))$.

Cor: Assuming the 3SUM conjecture, for any $0 < \gamma < 1$, any data structure for Set Disjointness has

$$t_p + N^{\frac{1+\gamma}{2-\gamma}} t_q = \Omega\left(N^{\frac{2}{2-\gamma} - o(1)}\right)$$

where N is the sum of the set sizes, t_p is the preprocessing time, and t_q is the time per query.

(From Thm: $N = \Theta(n^{2-\gamma} \log n)$)

Example: Data structures with constant query time

Make γ tend to 1, need $t_p = \Omega\left(N^{2-o(1)}\right)$

Evidence that *trivial preprocessing algorithm is optimal (for constant query)*

3SUM version

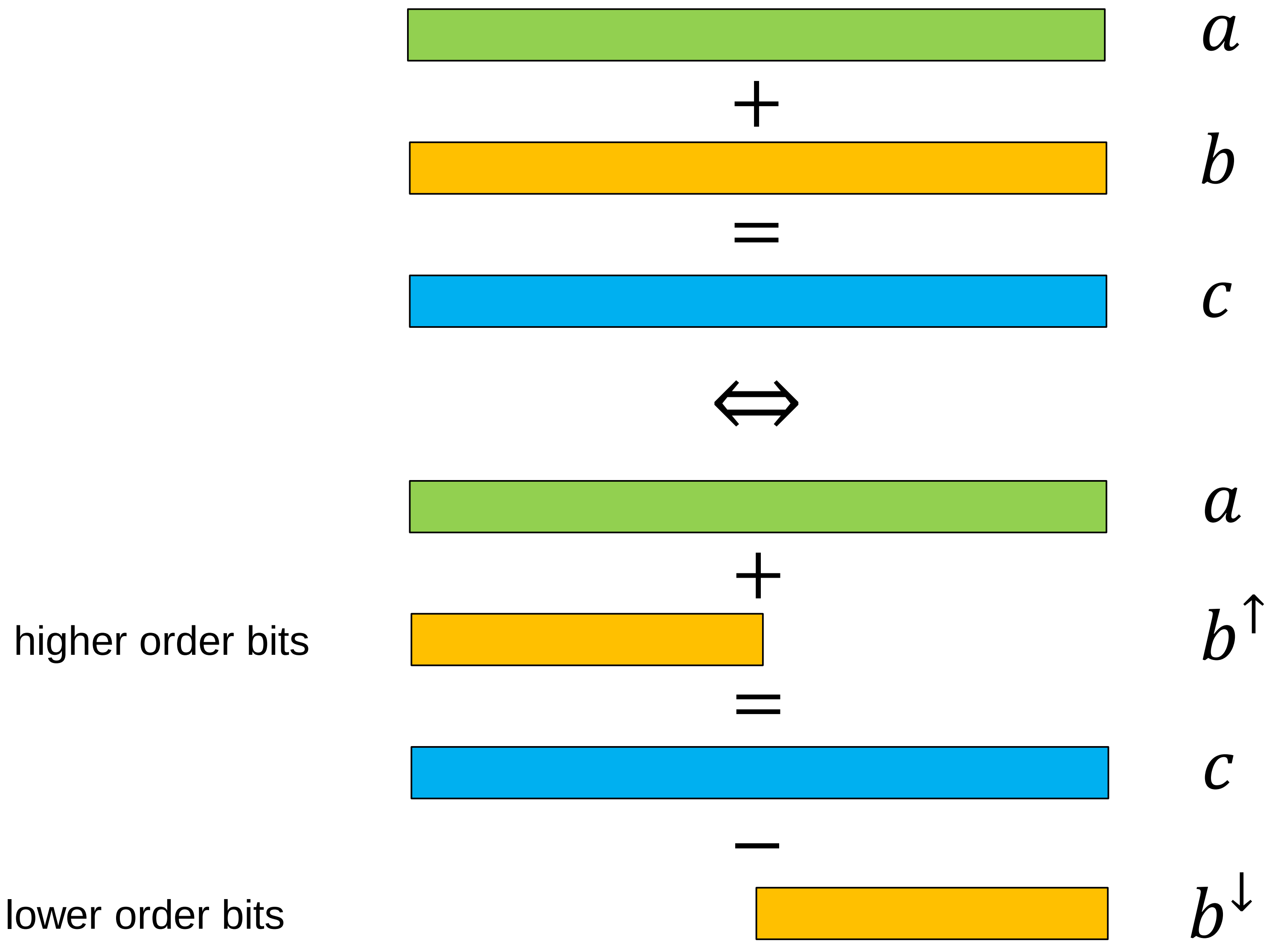
Given set $X \subseteq [U]$ of integers

are there $x, y, z \in X$ such that $x - y = z$?

In the following proof we use a balanced, linear hash function with uniform difference property. *(magically…)*

This can be modified for almost balanced, almost linear hash function with uniform difference property.

Crucial insight



Algorithm Overview

Set $R = n^\gamma$, $Q = \left(\frac{5n}{R}\right)^2$
Pick random hash functions $h: U \to [R]$ and $g_k: U \to [Q]$ for $k = 1$ to $10 \log n$

Initialize buckets $B[1], \ldots, B[R]$ s.t. $B[i] = \{x : h(x) = i\}$

For all $i \in [R], j \in \left[\sqrt{Q}\right]$, initialize buckets $B_k^{\uparrow}[i,j]$ and $B_k^{\downarrow}[i,j]$ s.t.

$$B_k^{\uparrow}[i,j] = \left\{g_k(x) + j \cdot \sqrt{Q} \pmod{Q} \mid x \in B[i]\right\}$$
$$B_k^{\downarrow}[i,j] = \{g_k(x) - j \pmod{Q} \mid x \in B[i]\}$$

Initialize k set intersection problems with $B_k^{\uparrow}[i,j]$'s and $B_k^{\downarrow}[i,j]$'s

For every $z \in X$ and every $i = 1$ to R
    Check if $B_k^{\uparrow}\left[i, g_k^{\uparrow}(z)\right]$ and $B_k^{\downarrow}\left[i - h(z) \pmod{R}, g_k^{\downarrow}(z)\right]$ intersect
    If intersection for all k:
        Search for $x \in B[i]$ and $y \in B[i - h(z) \pmod{R}]$ s.t.
        $x - y = z$ and output it if found

If nothing found: output 'no 3SUM'

$g_k^{\uparrow}(z)$: higher order bits of $g_k(z)$
$g_k^{\downarrow}(z)$: higher order bits of $g_k(z)$

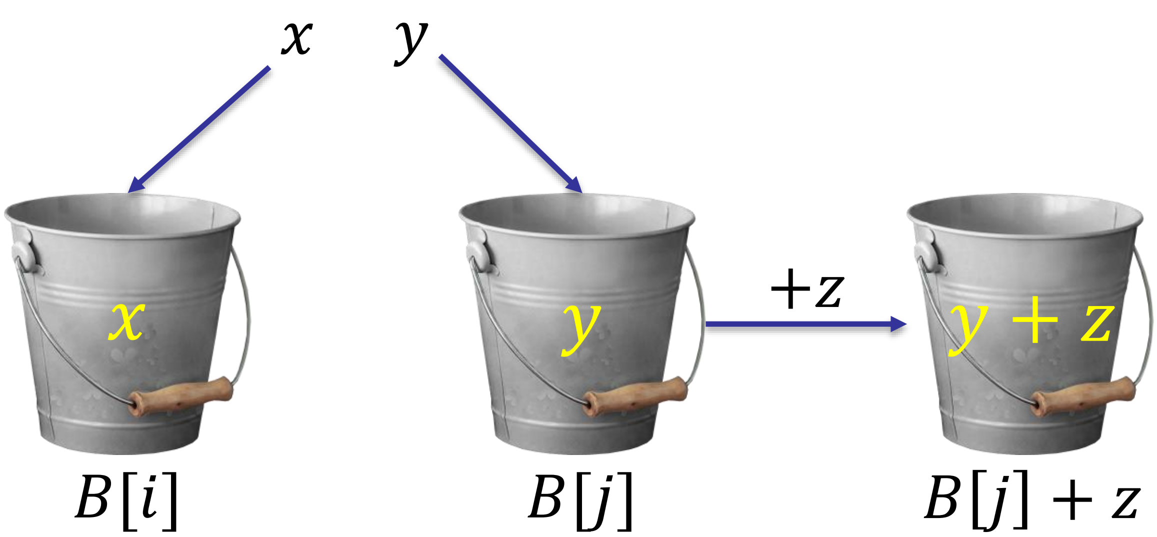
Correctness I

Algorithm verifies every triple before stopping
Need to show: if $x - y = z$, then algorithm finds it

Claim 1: If $x - y = z$, then $B[i] \cap (B[j] + z) \neq \emptyset$
where $i = h(x), j = i - h(z) \pmod R$

Linear hash function: $h(x) - h(y) = h(x - y) = h(z) \pmod R$
Thus: $j = h(x) - h(z) = i - h(z) \pmod R$

$$y \in B[j] \Rightarrow x = y + z \in B[j] + z$$



Correctness II

Claim 1: If $x - y = z$, then $B[i] \cap (B[j] + z) \neq \emptyset$
where $i = h(x), j = i - h(z) \pmod R$

Claim 2: If $B[i] \cap B[j] + z \neq \emptyset$, then $B^{\uparrow}\left[i, g_k^{\uparrow}(z)\right] \cap B^{\downarrow}\left[j, g_k^{\downarrow}(z)\right] \neq \emptyset\ \forall k$.

$$B[i] \cap B[j] + z \neq \emptyset$$
$$\Downarrow$$
$$g_k(B[i]) \cap g_k(B[j] + z) \neq \emptyset$$
$$\Updownarrow$$
$$g_k(B[i]) \cap \left(g_k(B[j]) + g_k(z)\right) \neq \emptyset$$
$$\Updownarrow$$
$$g_k(B[i]) \cap \left(g_k(B[j]) + g_k^{\uparrow}(z) + g_k^{\downarrow}(z)\right) \neq \emptyset$$
$$\Updownarrow$$
$$\left(g_k(B[i]) - g_k^{\uparrow}(z)\right) \cap \left(g_k(B[j]) + g_k^{\downarrow}(z)\right) \neq \emptyset$$
$$\Updownarrow$$
$$B^{\uparrow}\left[i, g_k^{\uparrow}(z)\right] \cap B^{\downarrow}\left[j, g_k^{\downarrow}(z)\right] \neq \emptyset$$

Conclusion: If $x - y = z$, then $B^{\uparrow}\left[i, g_k^{\uparrow}(z)\right] \cap B^{\downarrow}\left[j, g_k^{\downarrow}(z)\right] \neq \emptyset\ \forall k$.

Running time

Set intersection instance:

- Number of sets: $O(R\sqrt{Q}k) = O(n \log n)$
- Number of elements in each set: $O(\sqrt{Q}) = O(n^{1-\gamma})$
- Size of universe: $O(Q) = O(n^{2-2\gamma})$
- Number of set intersection queries: $O(nRk) = O(n^{1+\gamma} \log n)$

Finding witnesses:

- If $B_k^{\uparrow}[i, g_k^{\uparrow}(z)]$ and $B_k^{\downarrow}[j, g_k^{\downarrow}(z)]$ intersect, try to find $x \in B[i], y \in B[j]$ s.t. $x - y = z$
- Time $O\left(\frac{n}{R}\right)$ per witness check
- But: pair i, j could be **false positive** with no such $x \in B[i], y \in B[j]$
- Probability of false positive is small
- In expectation: $O(1)$ false positives (next slide)
- Total time: $O\left(\frac{n}{R} + (\#\text{false positives})\frac{n}{R}\right) = O\left(\frac{n}{R}\right)$

Bounding number of false positive

For a fixed z and any pair $x, y \in U$ s.t. $x - y \neq z$:

$$\Pr[g_k(x) = g_k(y) + g_k(z)] = \Pr[g_k(x - y) = g_k(z)] = \frac{1}{Q}$$

(linear and uniform difference)

Remember: Every bucket has size $\leq \frac{3n}{R}$ *(balanced)*

Prob. of false positive in buckets $B[i]$ and $B[j]$ for hash function g_k:

$$\Pr[g_k(B[i]) = g_k(B[j]) + g_k(z)] \leq \left(\frac{3n}{R}\right)^2 \frac{1}{Q} = \frac{9}{25}$$

Prob. of false positive in buckets $B[i]$ and $B[j]$ for **all** hash functions g_k:

$$\leq \frac{1}{n^c}$$

In expectation: total number of false positives is a constant.