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Randomized Algorithms and Probabilistic Analysis of Algorithms

Summer 2016

Exercise Set 3

Exercise 1: (5 Points)
We now consider the more general balls-into-bins setting with m balls being thrown to n bins, $n \neq m$. Show that the highest loaded bin contains $O(\frac{m}{n} + \log n)$ balls with high probability.

Exercise 2: (5 Points)
Consider a random variable $X = X_1 + \ldots + X_n$ such that each X_i is independent and identically distributed with $\mathbf{Pr}\left[X_i = 1\right] = p$, $\mathbf{Pr}\left[X_i = 0\right] = 1 - p$. To bound $\mathbf{Pr}\left[X \geq (1+\delta)\mathbf{E}\left[X\right]\right]$, you can use Markov's inequality, Chebyshev's inequality, and the Chernoff bound. State the resulting bounds in terms of n and p. For each of the three inequalities and each n, give an example value of p and δ such that its bound is the strongest of all three.

Exercise 3: (5 Points)
(Exercise 6.1. in Mitzenmacher/Upfal) Consider an instance of SAT with m clauses, where every clause has exactly k literals.

(1) Give a Las Vegas algorithm that finds an assignment satisfying at least $m(1-2^{-k})$ clauses, and analyze its expected running time.

(2) Give a derandomization of the randomized algorithm using the method of conditional expectations.

Exercise 4: (5 Points)
(Exercise 6.10 in Mitzenmacher/Upfal) A family $\mathcal{F}$ of subsets of $\{1, \ldots, n\}$ is an *antichain* if no set in $\mathcal{F}$ is properly contained in another set of $\mathcal{F}$.

(a) Give an example of an antichain of cardinality $\binom{n}{\lfloor n/2 \rfloor}$.

(b) Let f_k be the number of sets in $\mathcal{F}$ of size k. Show that

$$\sum_{0 \leq k \leq n} \frac{f_k}{\binom{n}{k}} \leq 1.$$

(Hint: Choose a random permutation of the numbers from 1 to n, and let $X_k = 1$ if the first k numbers in your permutation yield a set in $\mathcal{F}$. Let $X = \sum_{0 \leq k \leq n} X_k$. What can you say about X?)

(c) Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$ for every antichain $\mathcal{F}$.

Exercise 5: (5 Points)

(Exercise 6.16 in Mitzenmacher/Upfal) If

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1,$$

then it is possible to color the edges of K_n with two colors such that it has no monochromatic K_k subgraph. Use the Lovasz local lemma.