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Randomized Algorithms and Probabilistic Analysis of Algorithms

Summer 2016

Exercise Set 4

Exercise 1: (4 Points)

Consider the stochastic input model we used when bounding the number of Pareto-optimal solutions. For every ϕ, give an example of profits p_i and densities f_i such that there are $\Omega(\phi)$ Pareto-optimal solutions with probability 1.

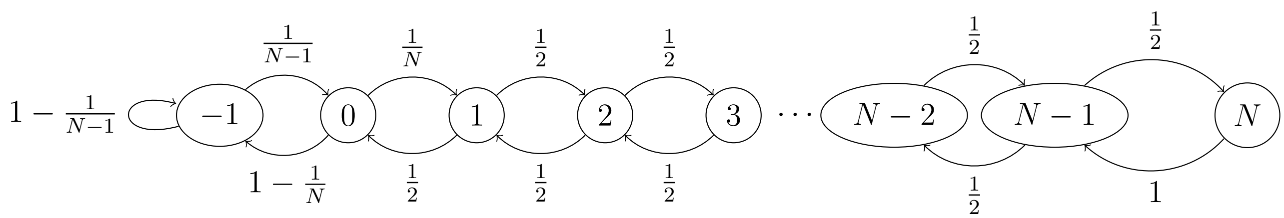
Exercise 2: (2+2 Points)

(a) Show that the cover time of the complete graph K_n is $\Theta(n \log n)$.
(Hint: You don't need use results from the current lecture but from another one.)

(b) Give an example of a Markov chain that is finite and aperiodic but has multiple stationary distributions.

Exercise 3: (4+2+2 Points)

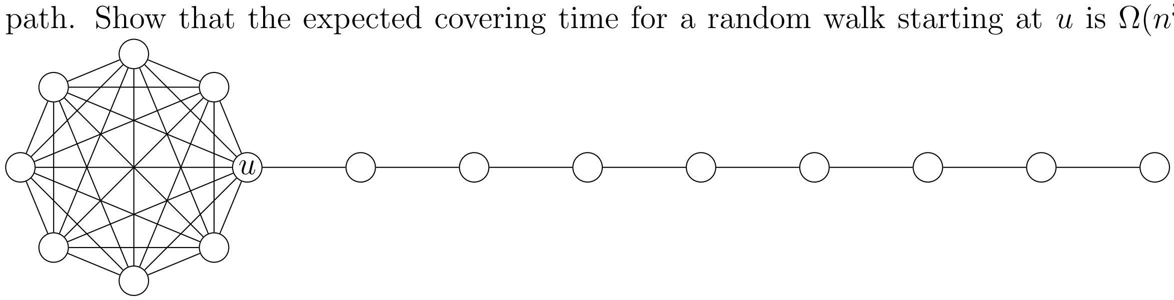
Consider the Markov chain on states $\{-1, 0, 1, \ldots, N\}$ defined by the arc weights in the following directed graph.



(a) What is $h_{i,i+1}$? (Hint: Find an expression for $h_{-1,0}$, then a recursive one for $h_{0,1}$ and so on. The result is in $\Theta(N)$ for $i = -1$ and in $\Theta(N^2)$ for all $i \geq 0$.)

(b) What is $h_{0,N}$? (Hint: Use linearity of expectation. The result is in $\Theta(N^3)$.)

(c) The lollipop graph on n vertices is a clique on $\frac{n}{2}$ vertices connected to a path on $\frac{n}{2}$ vertices, as shown below for $n = 8$. The node u is the point of the clique that is connected to the path. Show that the expected covering time for a random walk starting at u is $\Omega(n^3)$.



Exercise 4: (4 Points)
A random walk on a directed graph is defined exactly the same way as on an undirected one. In each step, the particle moves to one the neighbors of the current vertex, drawing one outgoing edge uniformly at random. Consider such a random walk on the following direction graph with vertices $\{0, 1, \ldots, n\}$. Each vertex has two directed edges: one that returns to 0 and one that moves to the vertex with the next higher number (with a self-loop at vertex n). Find the stationary distribution of this Markov chain. What is the cover time? (inspired by Exercise 7.21 in Mitzenmacher/Upfal)