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Randomized Algorithms and Probabilistic Analysis of Algorithms

Summer 2016

Exercise Set 6

Exercise 1: (1+5 Points)
We consider the *generalized assignment problem* (GAP), a generalization of knapsack and max-weight bipartite matching. We have n items and m kinds of bins. Bin i has a capacity of $t_i \geq 2$. When item j is placed in bin i, it consumes $w_{i,j} \in [0, 1]$ of the capacity but gives us a profit of $p_{i,j}$. The task is to assign the items to bins so as to maximize the profit. Items may also be assigned to no bin.

(a) State GAP as an integer program.

(b) Devise an algorithm based on randomized rounding to find an solution to GAP that is within a constant factor of the optimal solution to the LP relaxation.
Hints: Assume that you are given a fractional solution x^* to the LP relaxation. Use a scaled version of $(x^*_{i,j})_{i\in[n]}$ to decide how to assign item j. Use Markov's inequality to show that each constraint is fulfilled with constant probability. There is no need for a union bound this time.

Exercise 2: (3 Points)
Let us consider the minimization variant to the selection problem. An online algorithm is now α-competitive if for every sequence of costs $c_1, \ldots, c_n$. we have $\mathbf{E}\left[c(\text{ALG})\right] \leq \alpha \min_i c_i$. Show that there is no α-competitive algorithm for any finite α, not even a randomized one. (Hint: Observation 12.2)

Exercise 3: (2+2+4 Points)
Consider the following multiple-choice selection problem. An algorithm is presented a sequence of numbers $v_1, \ldots, v_n$. It may select a subset $ALG \subseteq [n]$ of size up to k, where k is a parameter. In class, we covered the case $k = 1$. In the following, we assume that an adversary defines both the values and the order. An algorithm is called α-competitive if for every choice of the adversary $\mathbf{E}\left[\sum_{i\in ALG} v_i\right] \geq \alpha \max_{S\subseteq[n],|S|\leq k} \sum_{i\in S} v_i$. Assume that n is known to the algorithm.

(a) Show that every deterministic algorithm is 0-competitive if $k < n$.

(b) Give a $\frac{k}{n}$-competitive randomized algorithm.

(c) Show that there is no α-competitive randomized algorithm for $\alpha > \frac{k}{n}$.