Exercise 8: Sort it out!

Task 1: Associative or not Associative?

a) Provide the truth table of $\diamond_{\mathrm{M}}$.

b) Show that $\diamond_{\mathrm{M}}$ is associative. Using a computer program is fine, as long as you provide clear and well-documented code and structure the output in a readable form.

c) Show that there is an associative operator $\{0,1\}^2 \to \{0,1\}$ whose closure is not associative! (Hint: There are not too many candidates, and quite a few can be ruled out easily. However, using a computer makes the search trivial!)

Solution

a)

$\diamond_{\mathtt{M}}$	00	0M	01	M1	11	1M	10	M0	MM
00	00	0M	01	M1	11	1M	10	M0	MM
0M	0M	0M	01	M1	M1	MM	MM	MM	MM
01	01	01	01	01	01	01	01	01	01
M1	M1	MM	MM	MM	0M	0M	01	M1	MM
11	11	1M	10	M0	00	0M	01	M1	MM
1M	1M	1M	10	M0	M0	MM	MM	MM	MM
10	10	10	10	10	10	10	10	10	10
M0	M0	MM	MM	MM	1M	1M	10	M0	MM
MM	MM	MM	MM	MM	MM	MM	MM	MM	MM

b) N/A

c) Binary addition is a counterexample, i.e., $(0\mathrm{M}+01)+01 = \mathrm{MM} \neq 1\mathrm{M} = 0\mathrm{M}+(01+01)$, yet addition is associative on stable values.

Task 2: Is it a CMUX?

a) Provide a small circuit implementing $\diamond_{\mathrm{M}}$. (Hint: If you can do c) right away, you can skip a), but you can score points for a) if c) proves challenging.)

b) Provide a small circuit implementing $\mathrm{out}_{\mathrm{M}}$. (Hint: As for a).)

c) Provide a small circuit that can be used to compute each bit of either function (changing to which wires the inputs go and negating inputs or outputs to the circuit is fine).

Solution

We solve all subtasks in one go by doing c). [See Figure 3 in the sorting paper on arxiv for the circuit : https://arxiv.org/pdf/1801.07549.pdf]

sel_1	sel_2	a	b	f
$\overline{b_1}$	$\overline{b_1}$	s_2	$\overline{s_1}$	$\overline{(\overline{s} \diamond_{\mathrm{M}} \overline{b})}_1$
b_2	b_2	s_2	$\overline{s_1}$	$\overline{(\overline{s} \diamond_{\mathrm{M}} \overline{b})}_2$
$\overline{s_1}$	s_2	b_1	b_2	$\mathrm{out}_{\mathrm{M}}(s,b)_1$
s_2	$\overline{s_1}$	b_2	b_1	$\mathrm{out}_{\mathrm{M}}(s,b)_2$

Task 3*: Too much Detail?

a) One can provide smaller implementations of a CMUX when working on the transistor level. Find out about this!

b) Can you provide efficient transistor-level implementations of the subcircuits implementing $\diamond_{\mathrm{M}}$ and $\mathrm{out}_{\mathrm{M}}$?

c) Induce a flow of information to your fellow students in the TA session!