Convex hulls in $\mathbb{R}^2$

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Computaional Geometry
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max planck institut
informatik

Overview

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- Problem definition

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- Computaitonal models, input and output

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- Naive algorithm

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- Chan's algorithm

Convex hull

Notations, definitions

$\mathbb{R}^d$ is d-dimensional Euclidean space

$P = \{p_1, \ldots, p_n\}$ set of n points

$X \subseteq \mathbb{R}^d$ is *convex* if for any $p, q \in X$ we have $pq \subseteq X$

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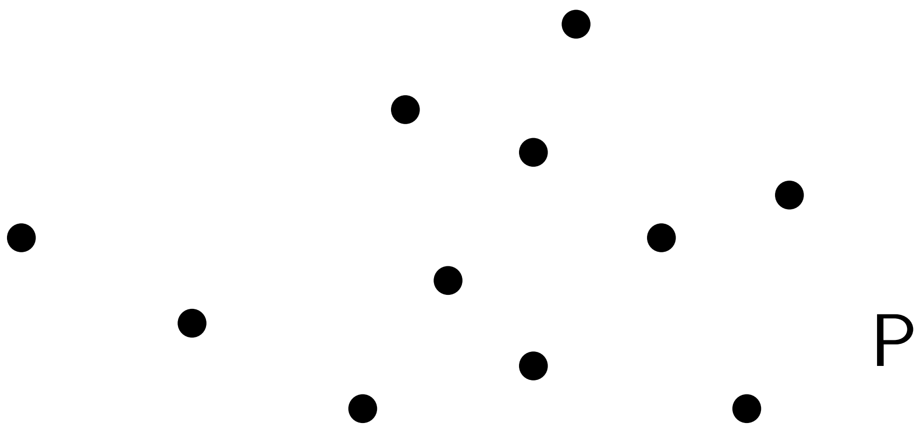

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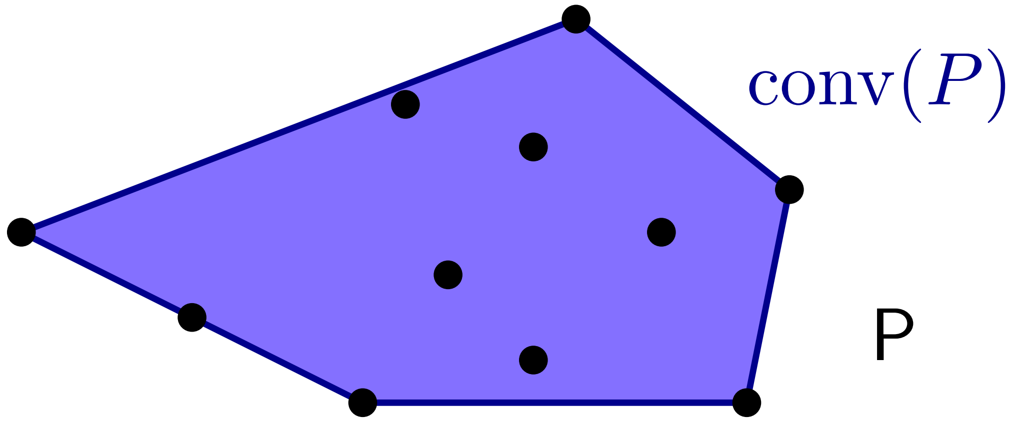

Real RAM vs. Word RAM

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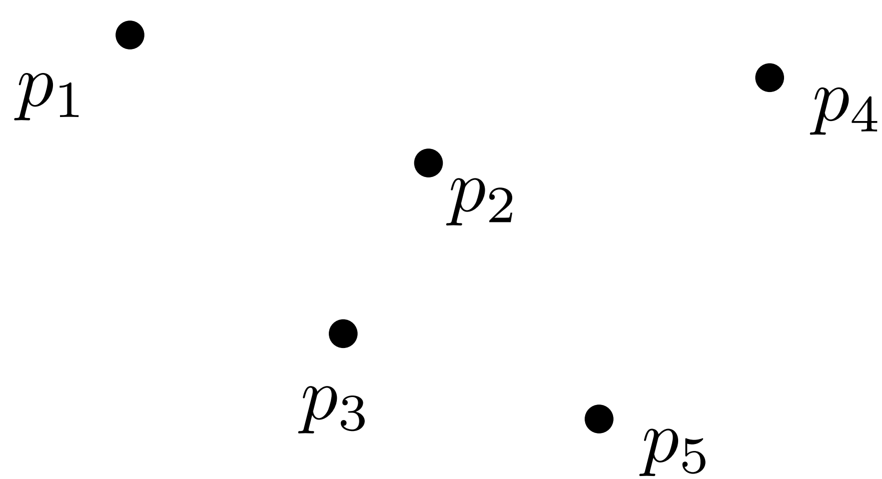
Real inputs and outputs, can extend with $\sqrt{.}, \ln(.)$	Exact arithmetic for rational inputs with $+ - * /$
Unrealistic power	Too restrictive?

Convex hull: input and output

Input: Points with coordianate pairs $(x, y) \in \mathbb{R}^2$

$$(e, \pi), (3, 3), (2.95, 2.9), (\sqrt{11}, 3.05), (\pi, e)$$



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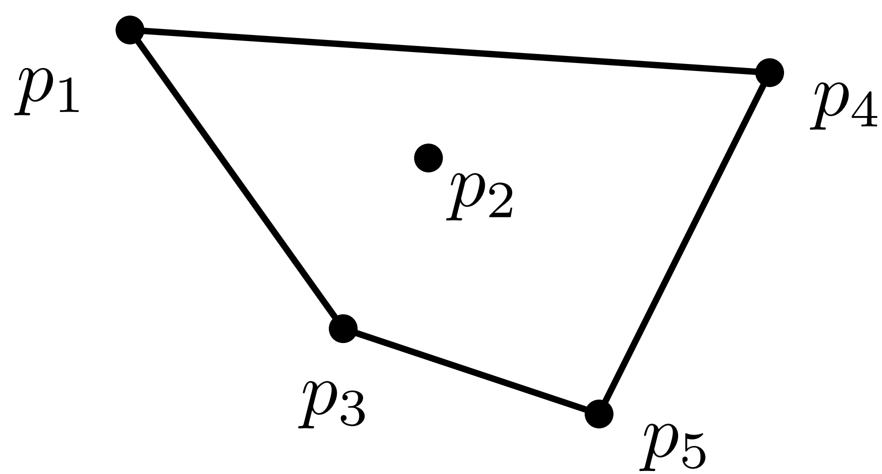
$$p_1, p_4, p_5, p_3$$



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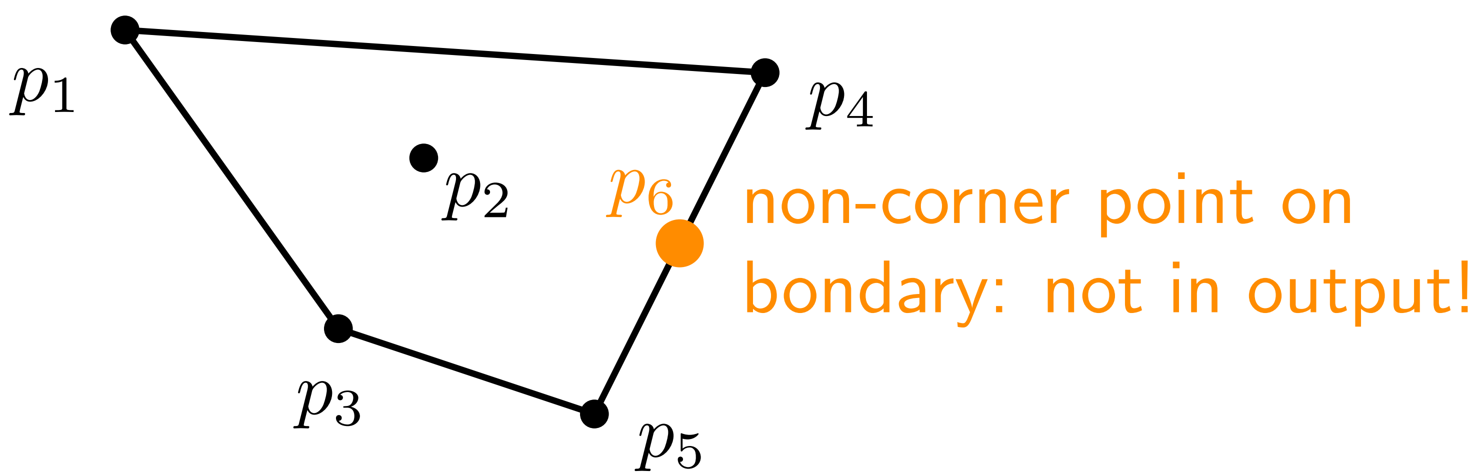

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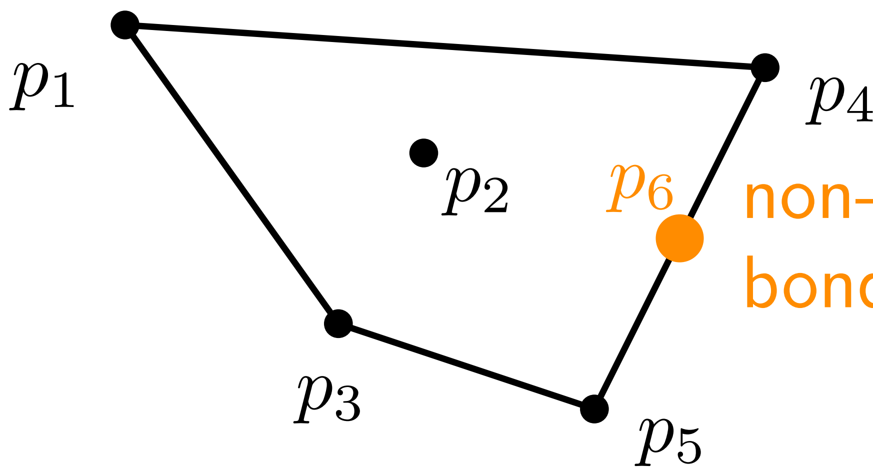

non-corner point on bondary: not in output!

Everything works with rational inputs on Word RAM!

Naive algorithm

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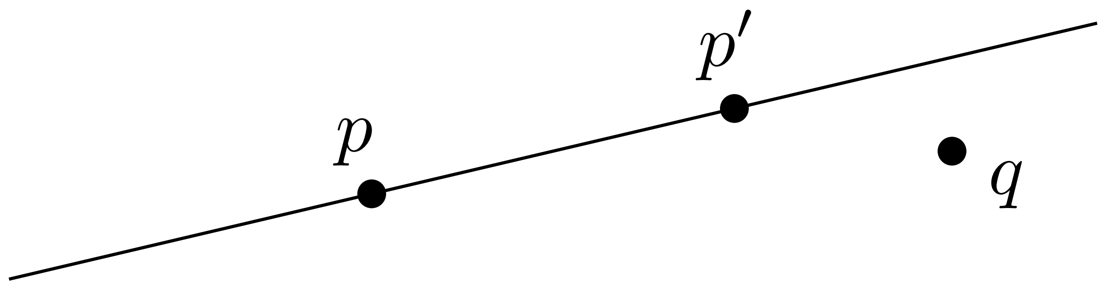

$$\operatorname{sgn}\left(\begin{vmatrix} p_x & p_y & 1 \\ p'_x & p'_y & 1 \\ q_x & q_y & 1 \end{vmatrix}\right)$$

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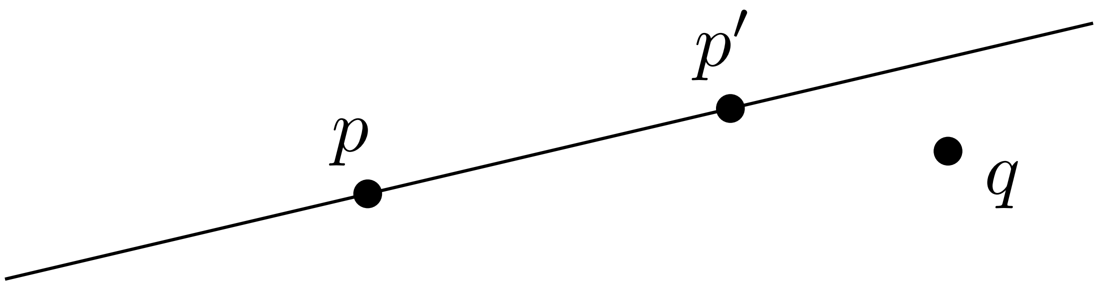

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Naive Convex Hull in $\mathbb{R}^2$

For each $p, p' \in P$,

check if all $q \in P \setminus \{p, p'\}$ is on the left of line pp'.

If yes, then p' follows p in $\operatorname{conv}(P)$.

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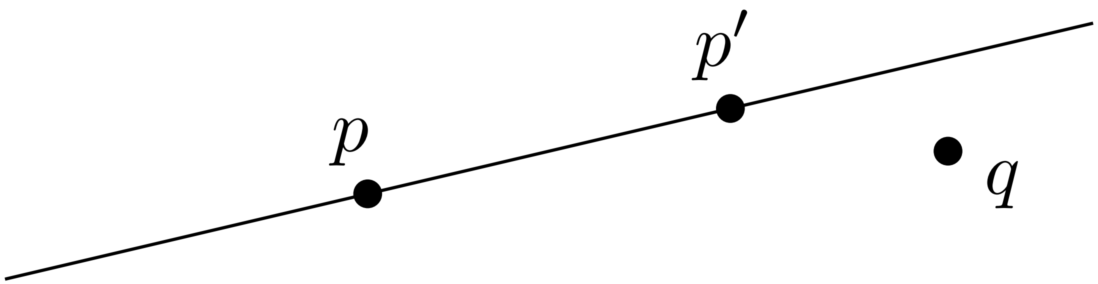

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Running time: $\binom{n}{2} \cdot (n-2) \cdot O(1) = O(n^3)$

Graham's scan (1972)

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Suppose points have distinct x-coordinates.

Let $p_1, \ldots, p_n$: points sorted with increasing x-coordinates.

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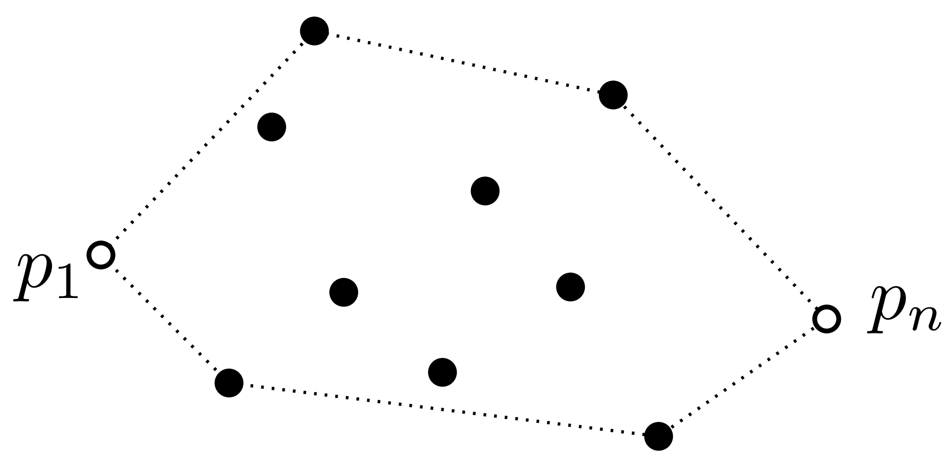

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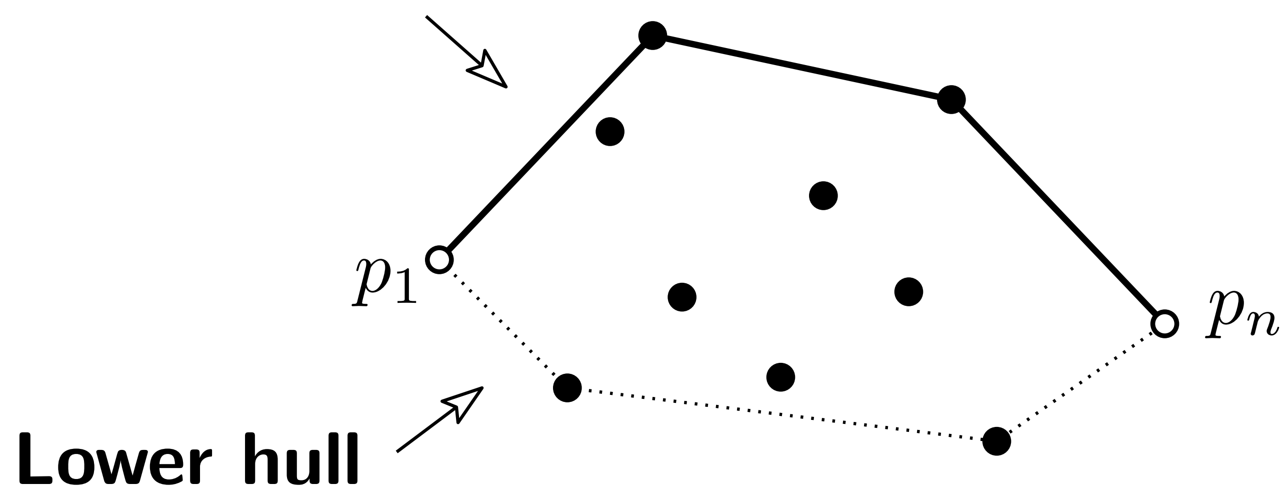
Upper hull
part of the hull after p_1 and before p_n in clockwise order



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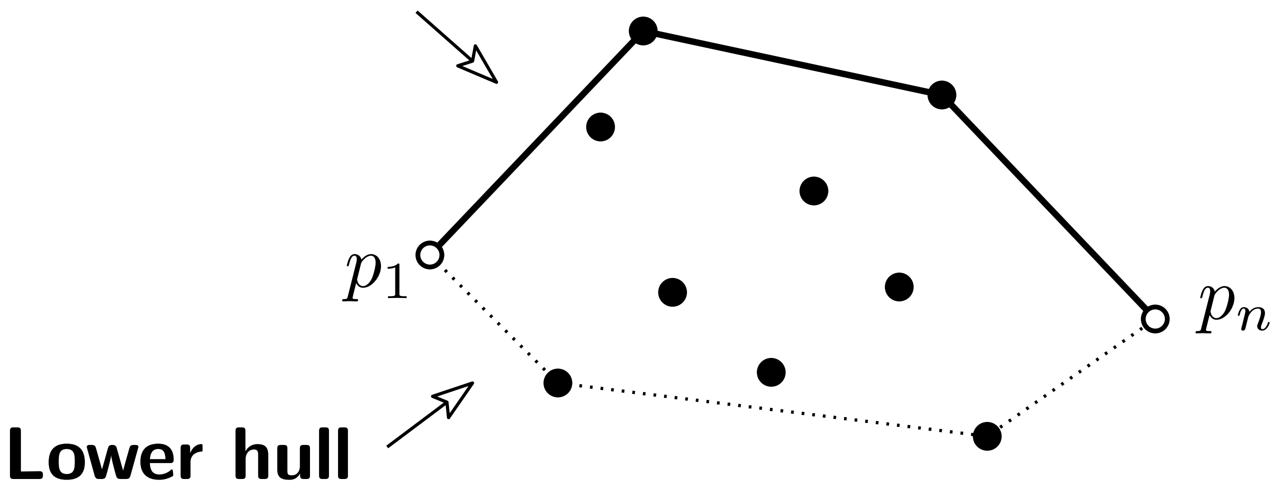
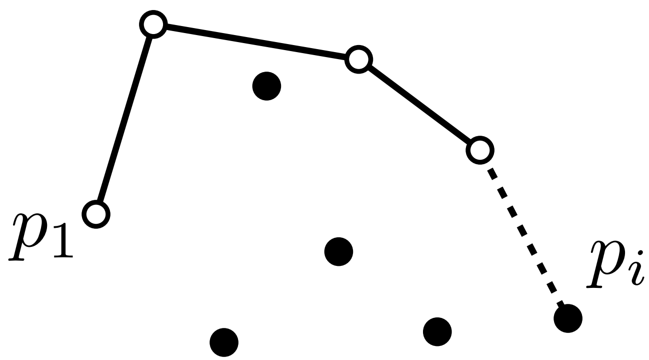


Idea:
Add points left to right, update upper hull after each addition

Graham's Scan: update

Right turn

(p_i is below last hull segment)



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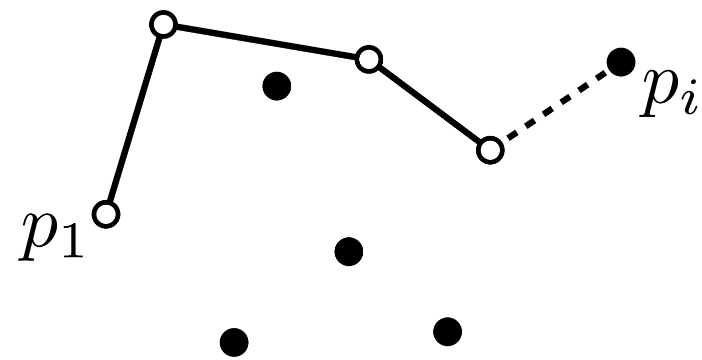

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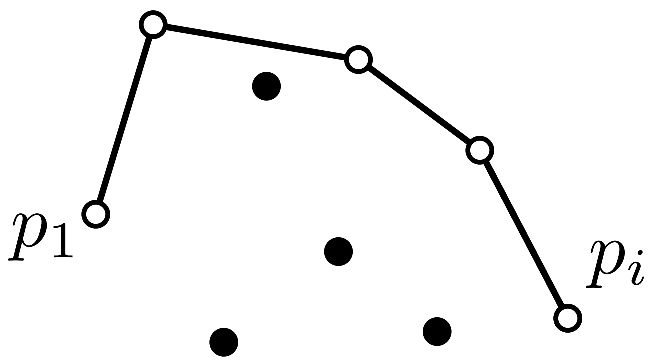

Add p_i to the upper hull

Left turn

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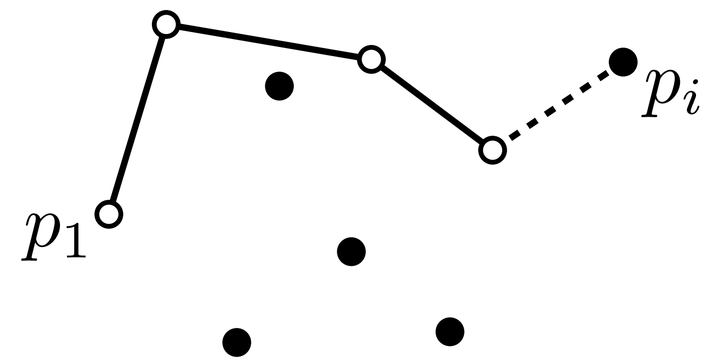

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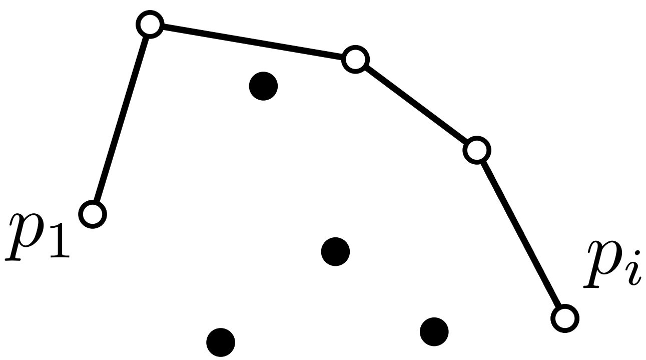

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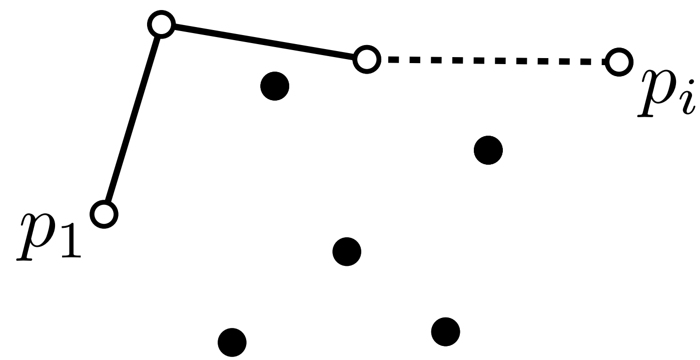

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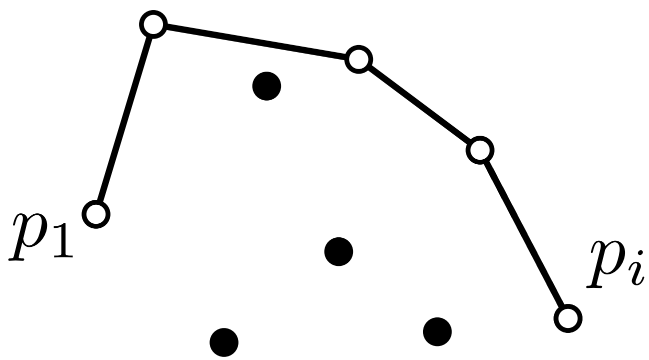

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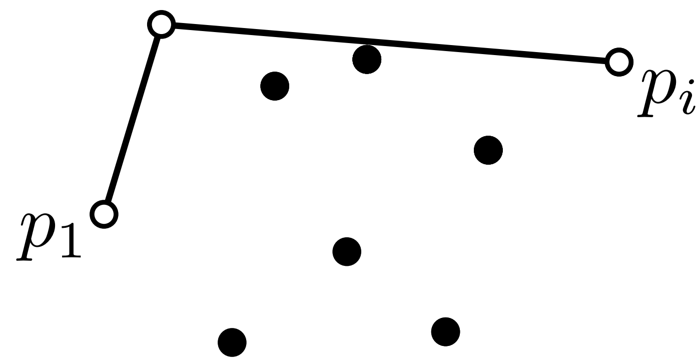

Add p_i but remove previous hull point until left turn disappears

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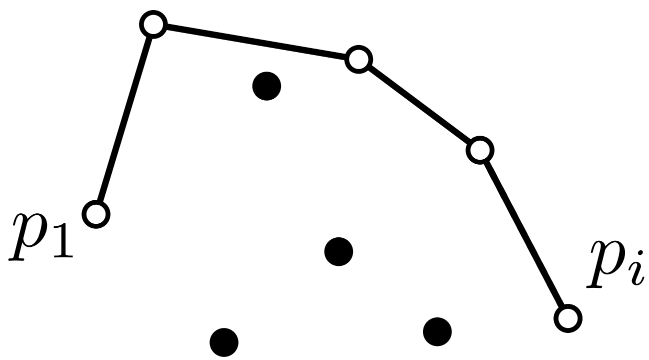

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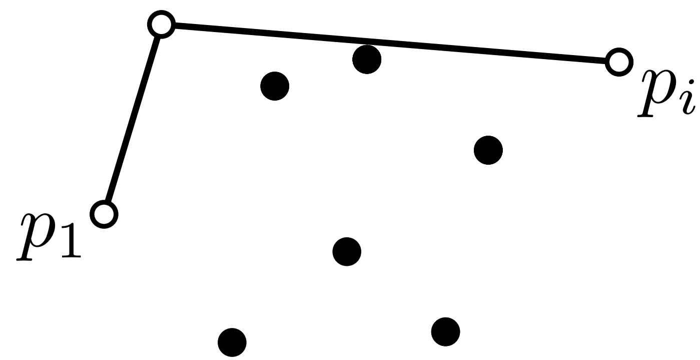

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Similalrly for lower hull, after adding p_i:

while last three points of lower hull q, q', p_i are a right turn:

remove the middle point q'

Graham's Scan: pseudocode + runtime

```
Sort P by increasing x-coordinates
Add p_1, p_2 to U and L
for i = 3 to n do
    Add p_i to U and L
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return L and reverse of U
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Each $p \in P$ is:

- added once to U (same for L) $\longrightarrow O(n)$
- removed at most once from U (same for L) $\longrightarrow O(n)$

Triplets checked in While loop heads $\longrightarrow O(n)$

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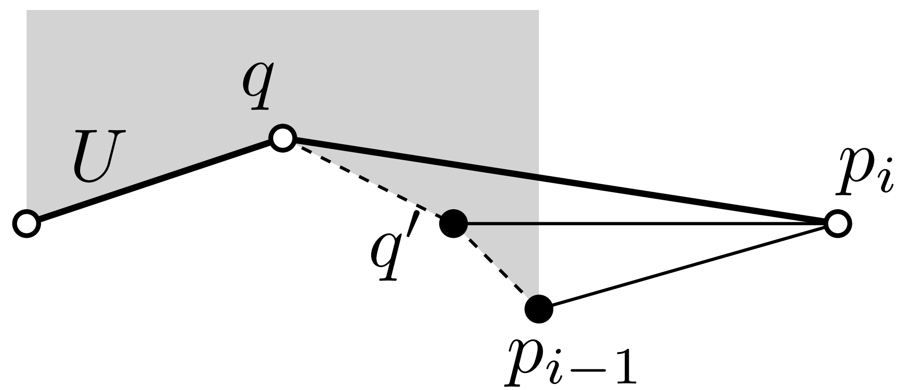
Suppose U is the upper hull of $p_1, \ldots, p_{i-1}$

$\Rightarrow$ Gray is empty

p_i is added to U ✓



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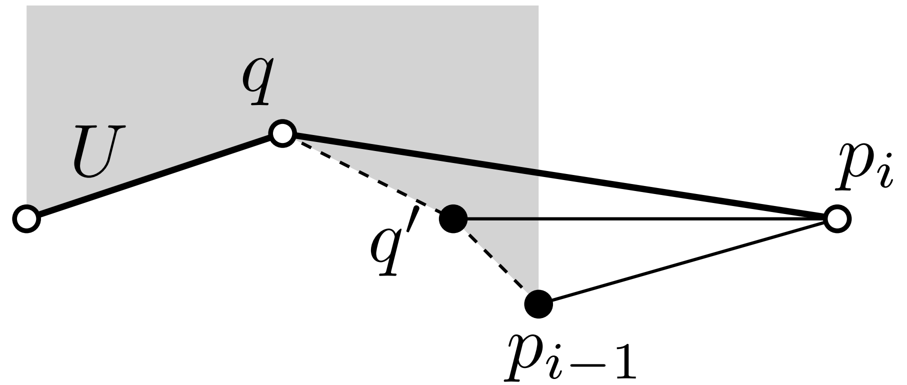
Similarly, q' is below qp_i
$\Rightarrow$ All deleted vertices are below the new U.



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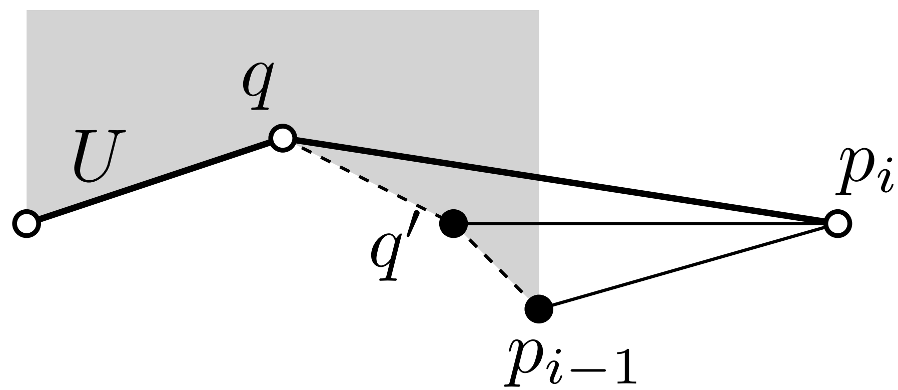
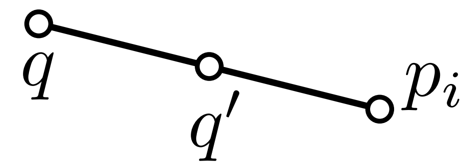


Graham's Scan: non-general position

- collinear triple:

Instead of deleting for left turn in U:
Delete if straight or left turn



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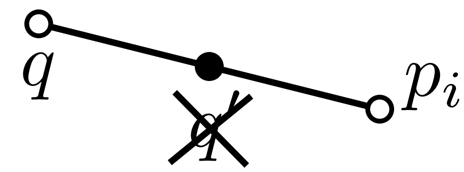

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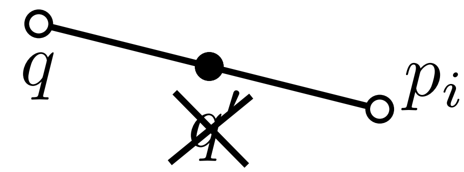

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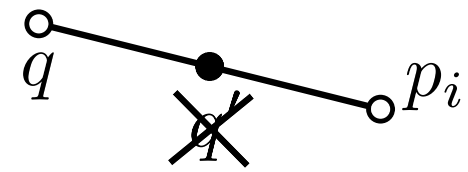

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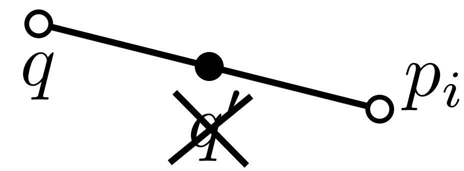

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p_1, p_n are still on the hull.
Upper hull U: part of hull after p_1 and before p_n in cw order

Practicalities

Collinear triples are common (grids!)

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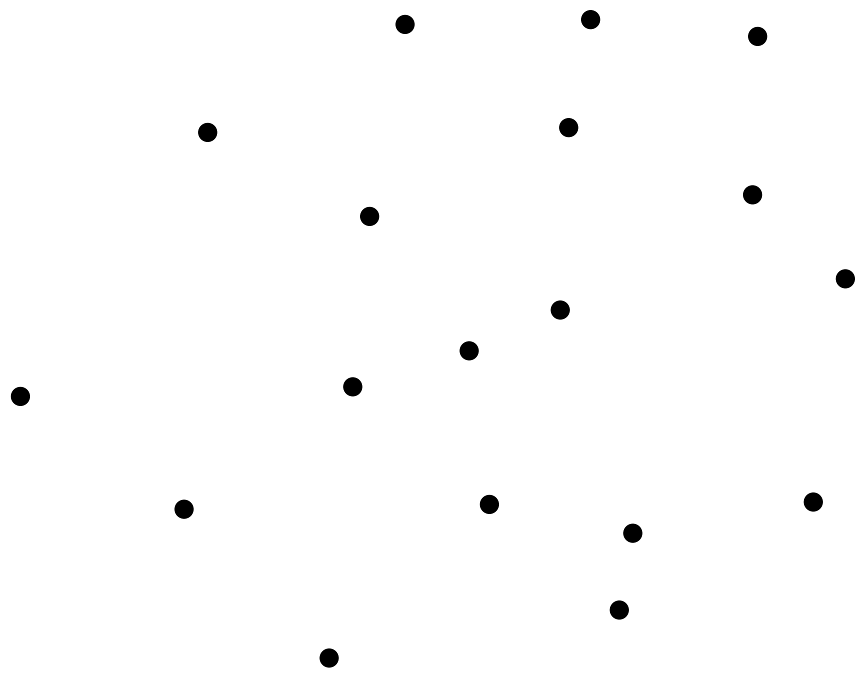
Almost collinear triples are also common!



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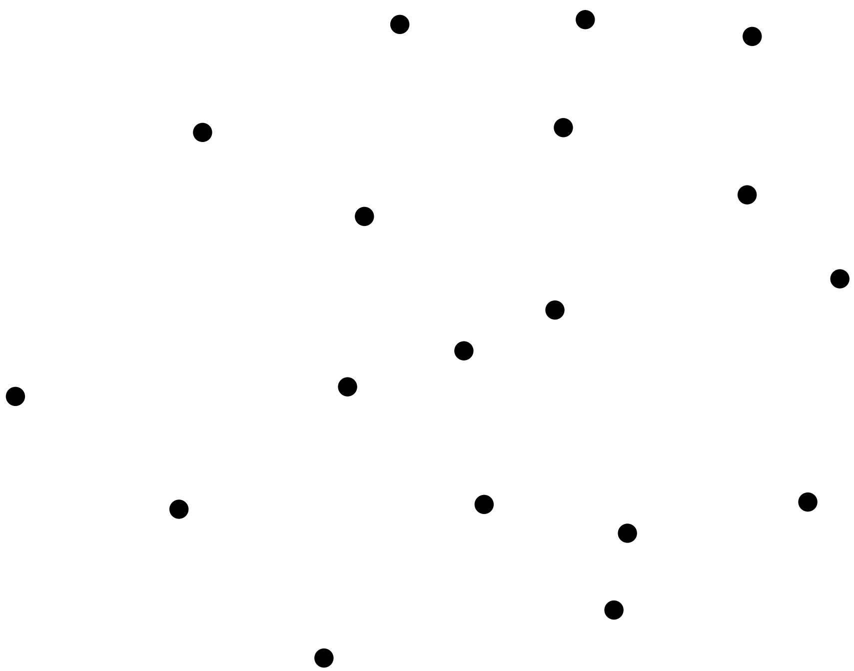

P uniform random in $[0,1]^2$:

$$\mathbb{E}(\max_{p,q,r \in P} \sphericalangle pqr) = \pi - O(1/n^3)$$

not peer-reviewed source

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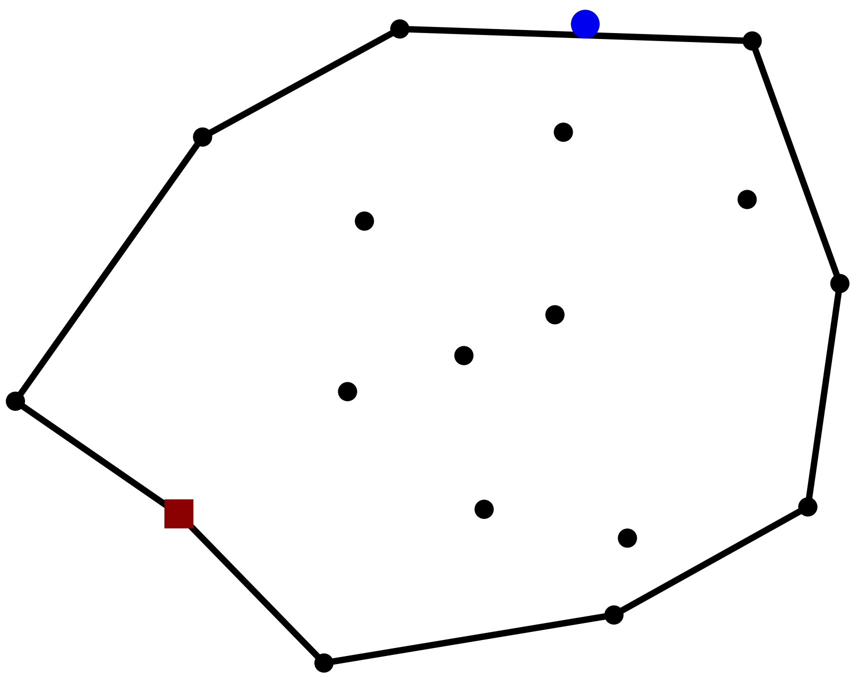

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Naive floating point implementation of Real RAM:
false positives and false negatives

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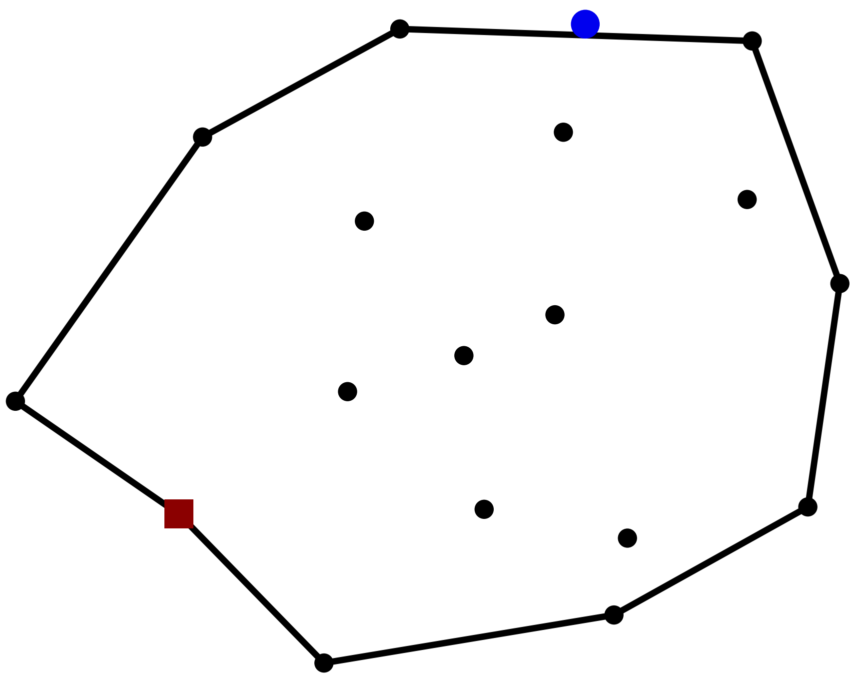

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Naive floating point implementation of Real RAM:
false positives and false negatives

Good software libraries (e.g. CGAL) can protect you.

Chan's algorithm (1996)

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$\rightarrow$ running time expressed as function of input **and output** size

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What about $O(n + h \log h)$?

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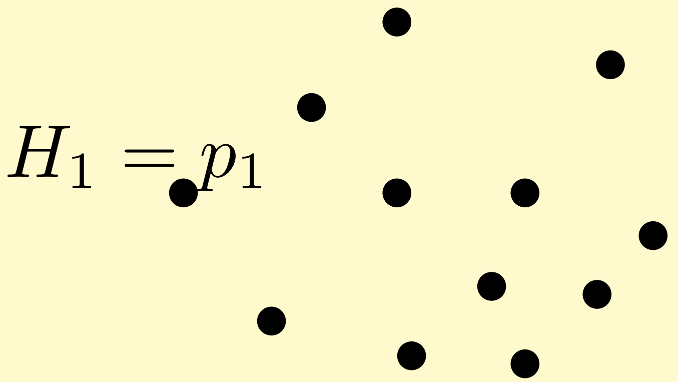

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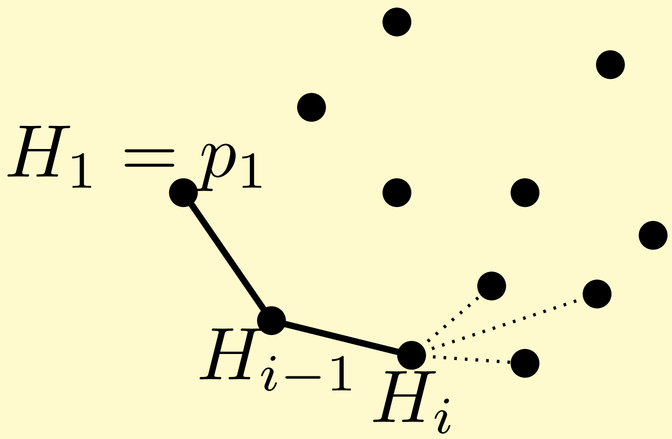

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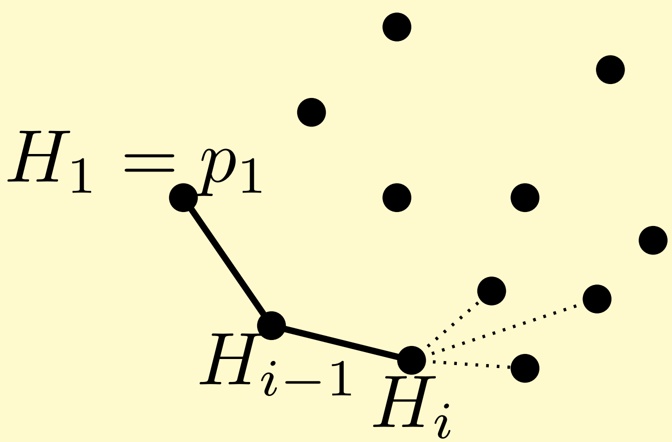
- Repeated h times $\rightarrow O(nh)$ time in total



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Let $m \in \{1, \ldots, n\}$ be a parameter.

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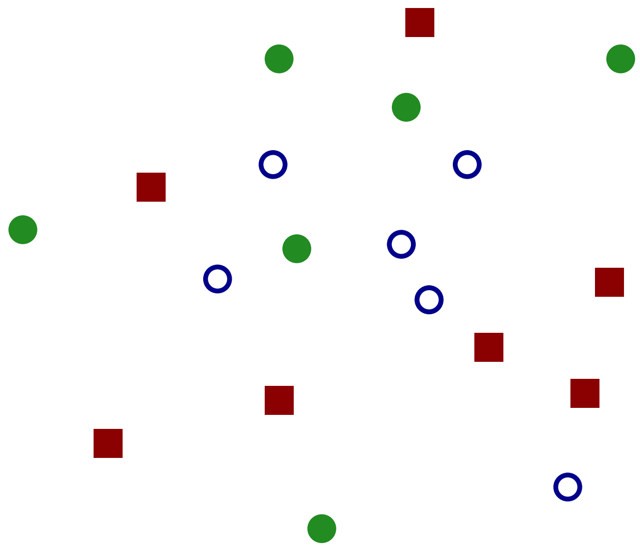

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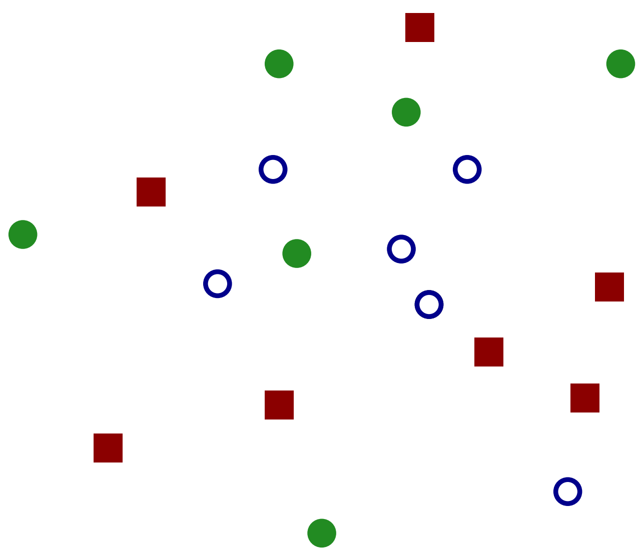

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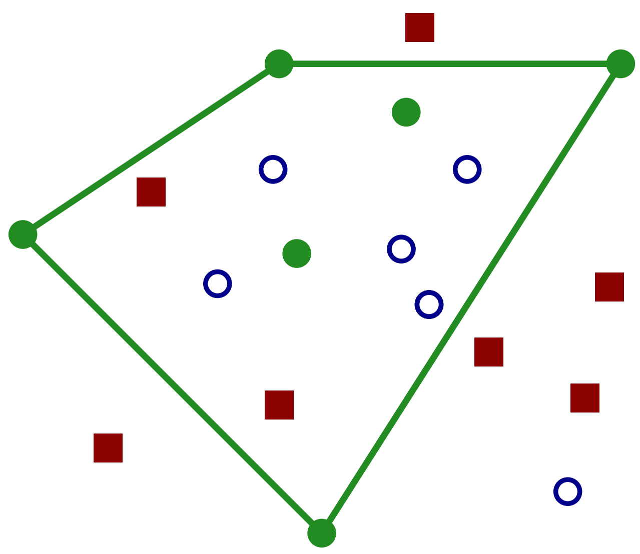

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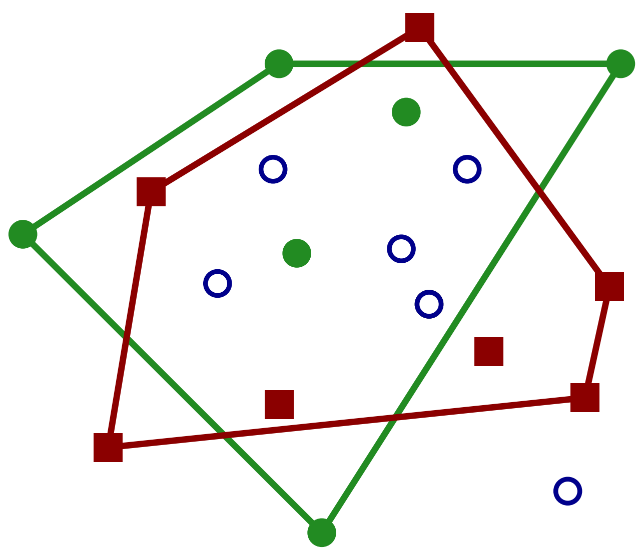

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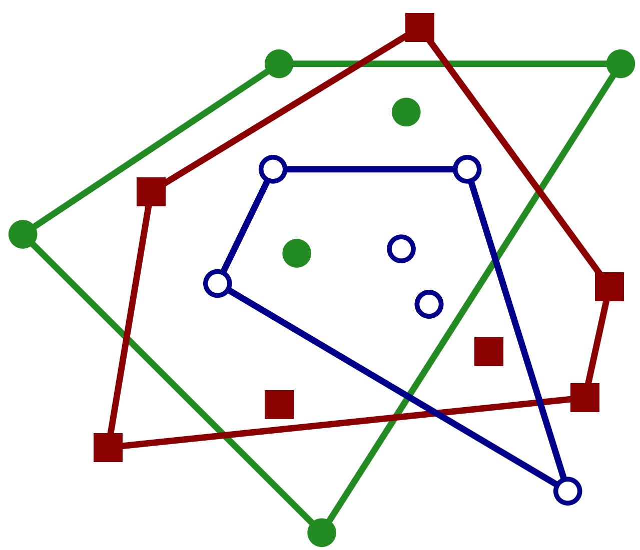

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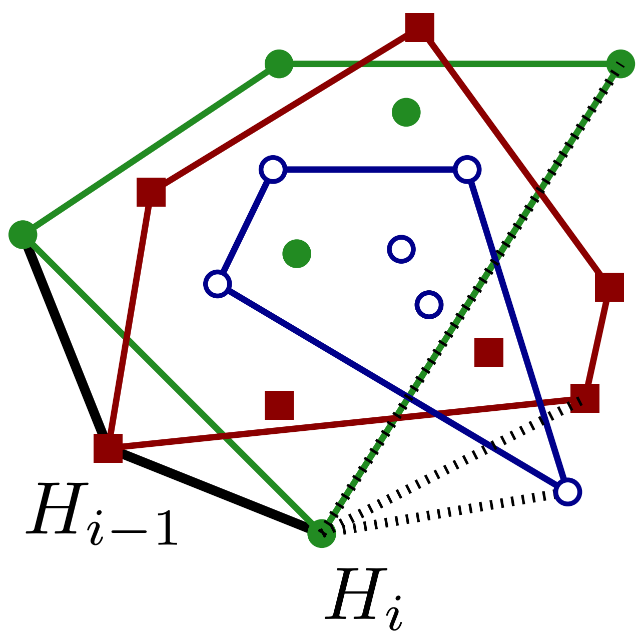

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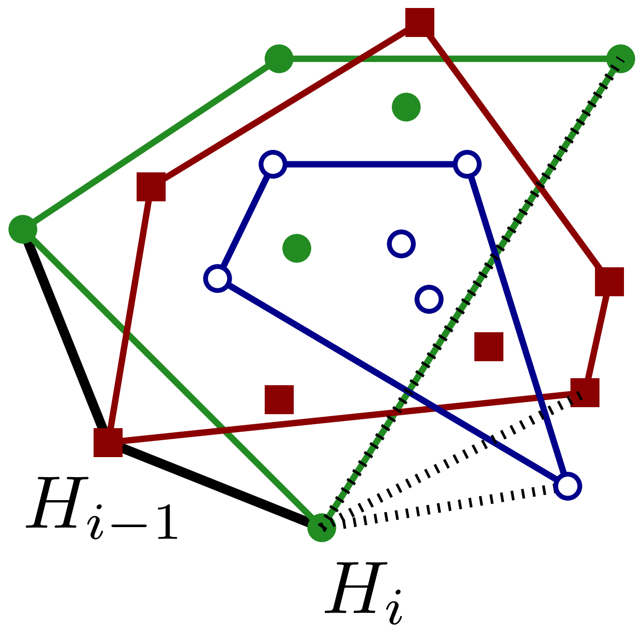

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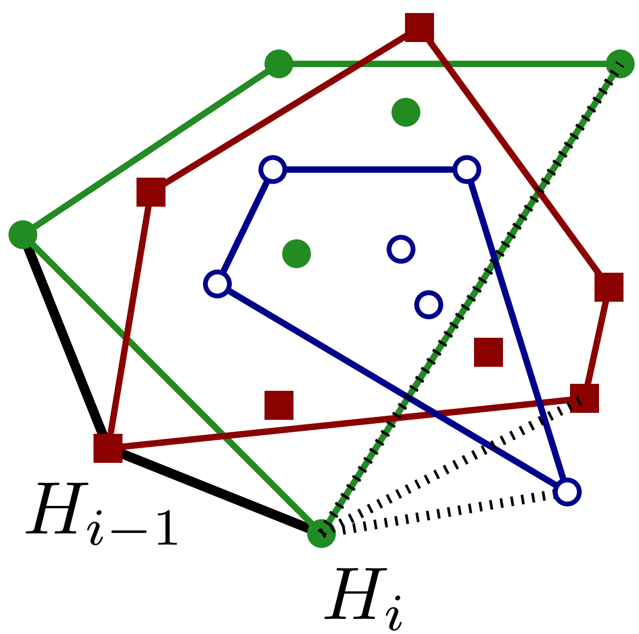

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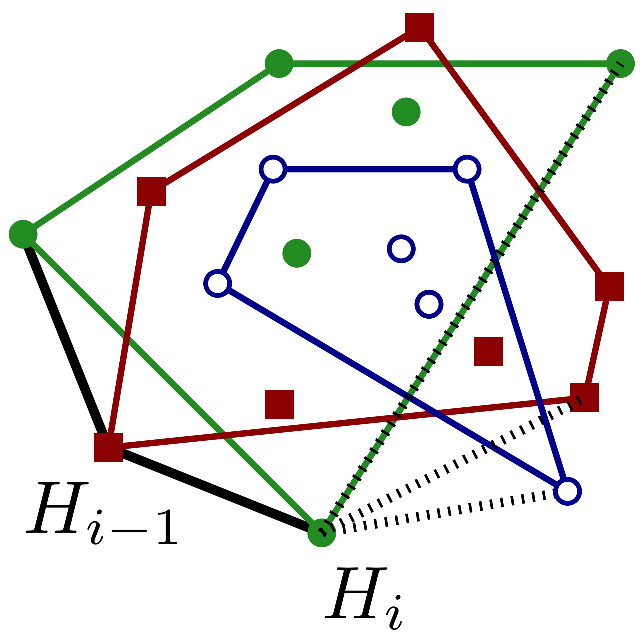

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Time to find tangent of $\text{conv}(P_j)$

Chan's algorithm: running time and tangent finding

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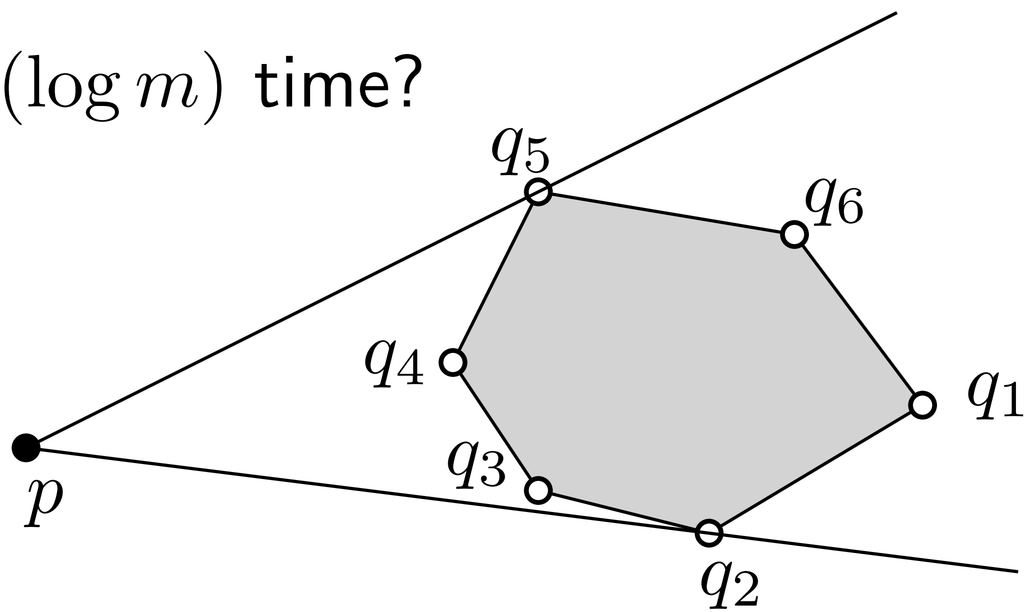
Tangent finding in $O(\log m)$ time?



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Idea 1: Let $m = 1, 2, \ldots, n$, run giftwrapping for m steps
Stop when gift is wrapped. (Then $m \geq h$ holds.)

$$\sum_{m=1}^{h} cn \log m = \Omega(hn)...$$

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Idea 2: Let $m = 2^1, 2^2, 2^3, \ldots, 2^{\log n}$, run wrapping for m steps

$$\sum_{i=1}^{\lceil \log h \rceil} cn \log 2^i = \sum_{i=1}^{\lceil \log h \rceil} cni = \Theta(n \log^2 h) \ldots$$

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Idea 3: Let $m = 2^{2^0}, 2^{2^1}, 2^{2^2}, \ldots, 2^{2^{\lceil \log \log n \rceil}}$, do m wrap-steps

$$\sum_{i=0}^{\lceil \log \log h \rceil} cn \log 2^{2^i} = \sum_{i=0}^{\lceil \log \log h \rceil} cn 2^i$$

$$= cn(2^{\lceil \log \log h \rceil + 1} - 1) = O(n \log h)$$

Chan's algorithm: Recap

for $i = 1$ to $\lceil \log \log n \rceil$ **do**
  $m = 2^{2^i}$
  for $j = 1$ to $\lceil n/m \rceil$ **do**
    Create group P_j
    $(q_1^j, q_2^j, \ldots,) = Graham(P_j)$
  H_1 =leftmost point in P
  for $s = 2$ to m **do**
    for $j = 1$ to $\lceil n/m \rceil$ **do**
      $q^j = TangentFind(H_{s-1}, (q_1^j, q_2^j, \ldots))$
    $H_s =$ point q^j maximizing $\sphericalangle H_{s-2} H_{s-1} q^j$
    if $H_s = H_1$ **then return** $(H_1, \ldots, H_{s-1})$