Convex hulls in $\mathbb{R}^3$

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Computaional Geometry
Summer semester 2020

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informatik

Overview

- Computaitonal model, input and output

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- Preparata–Hong divide&conquer algorithm (1977)

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- Chan's algorithm in $\mathbb{R}^3$ (1996)
- Higher-dimensional convex hulls

Convex hull: input and output

Input: Points with coordianate pairs $(x, y, z) \in \mathbb{R}^3$

$$(e, \pi, 1), (3, 3, \sqrt{5}), (2.95, 2.9, 2^{1.2}), (\sqrt{11}, 3.05, \sqrt[3]{3})$$

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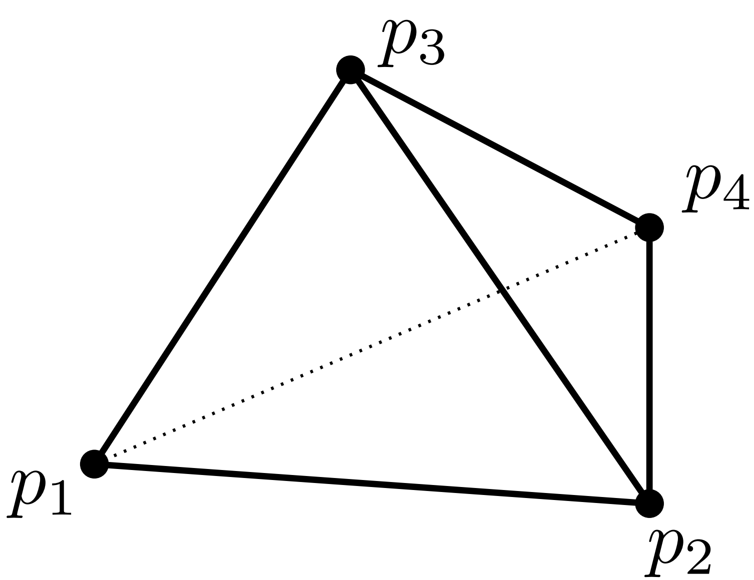

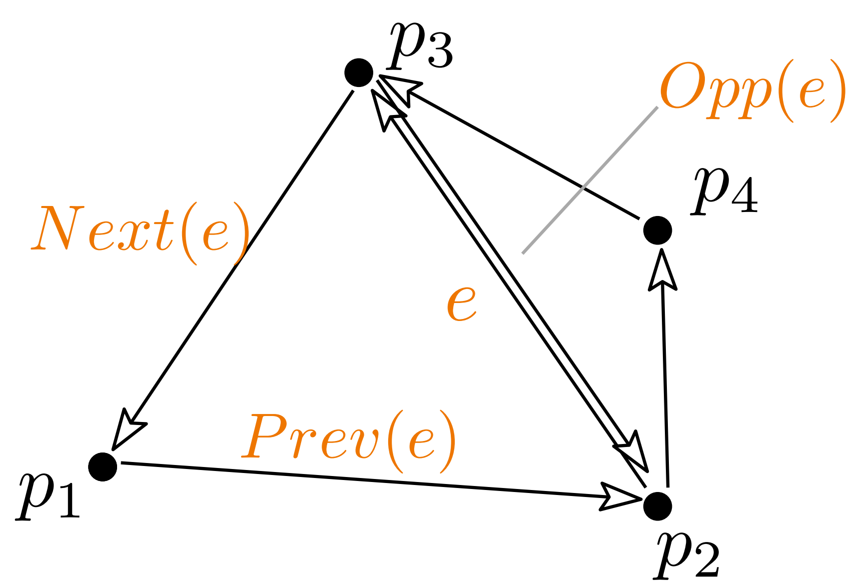

Doubly connected edge list, facets are ccw cycles from outside
arcs know: opposite, next, prev arc

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Total complexity (vertices+edges): $\leq 4h - 6 = O(h)$. □

Basic operation and naive approach

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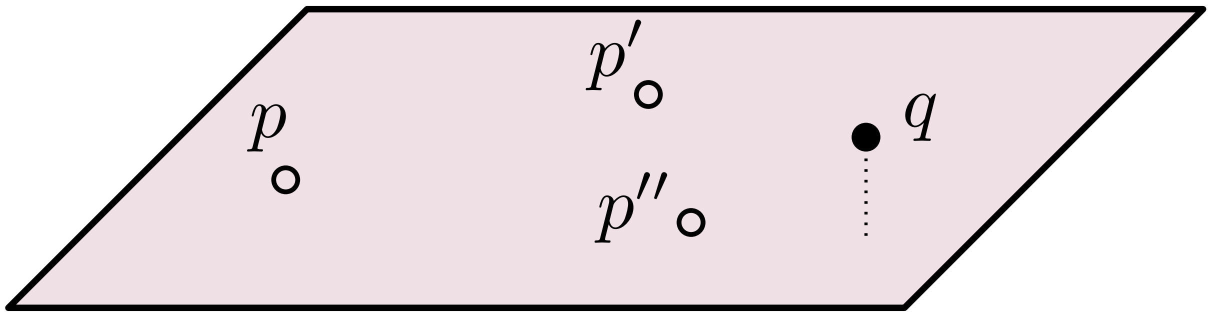

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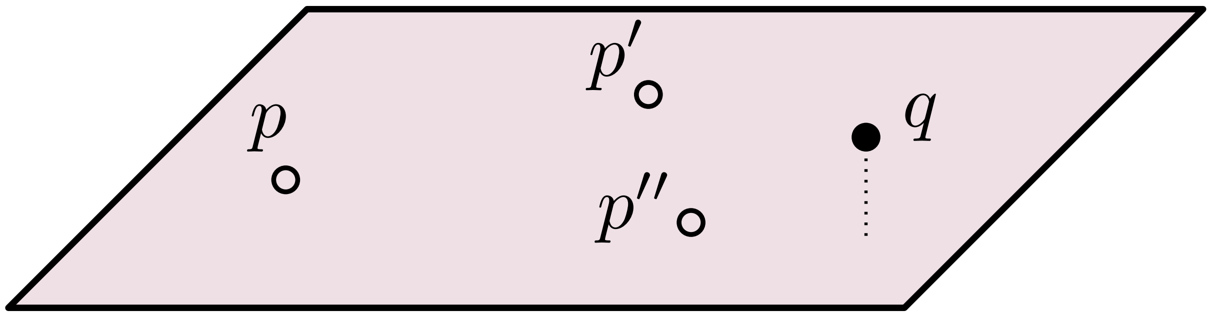

$$\begin{vmatrix} p_1 & p_2 & p_3 & 1 \\ p'_1 & p'_2 & p'_3 & 1 \\ p''_1 & p''_2 & p''_3 & 1 \\ q_1 & q_2 & q_3 & 1 \end{vmatrix}$$

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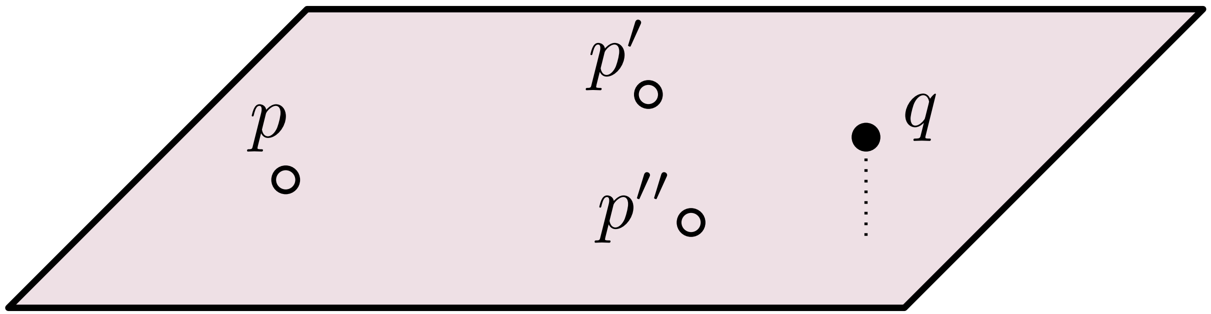

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Naive Convex Hull in $\mathbb{R}^3$

For each $p, p', p'' \in P$,

check if all $q \in P \setminus \{p, p', p''\}$ is on same side of plane $pp'p''$.

If yes, then $pp'p''$ is a face.

Running time: $\binom{n}{3} \cdot (n-3) \cdot O(1) = O(n^4)$

$\mathbb{R}^3$ convex hull with divide and conquer (Preparata–Hong, 1977)

Divide and conquer

$(p_1, \ldots, p_n) = \textsc{LexicographicSort}(P)$
$H = \textsc{Hull3dim}((p_1, \ldots, p_n))$

function $\textsc{Hull3dim}((p_1, \ldots, p_n))$
  if $n \leq 4$ **then**
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  $H = \textsc{Merge}(H_1, H_2)$
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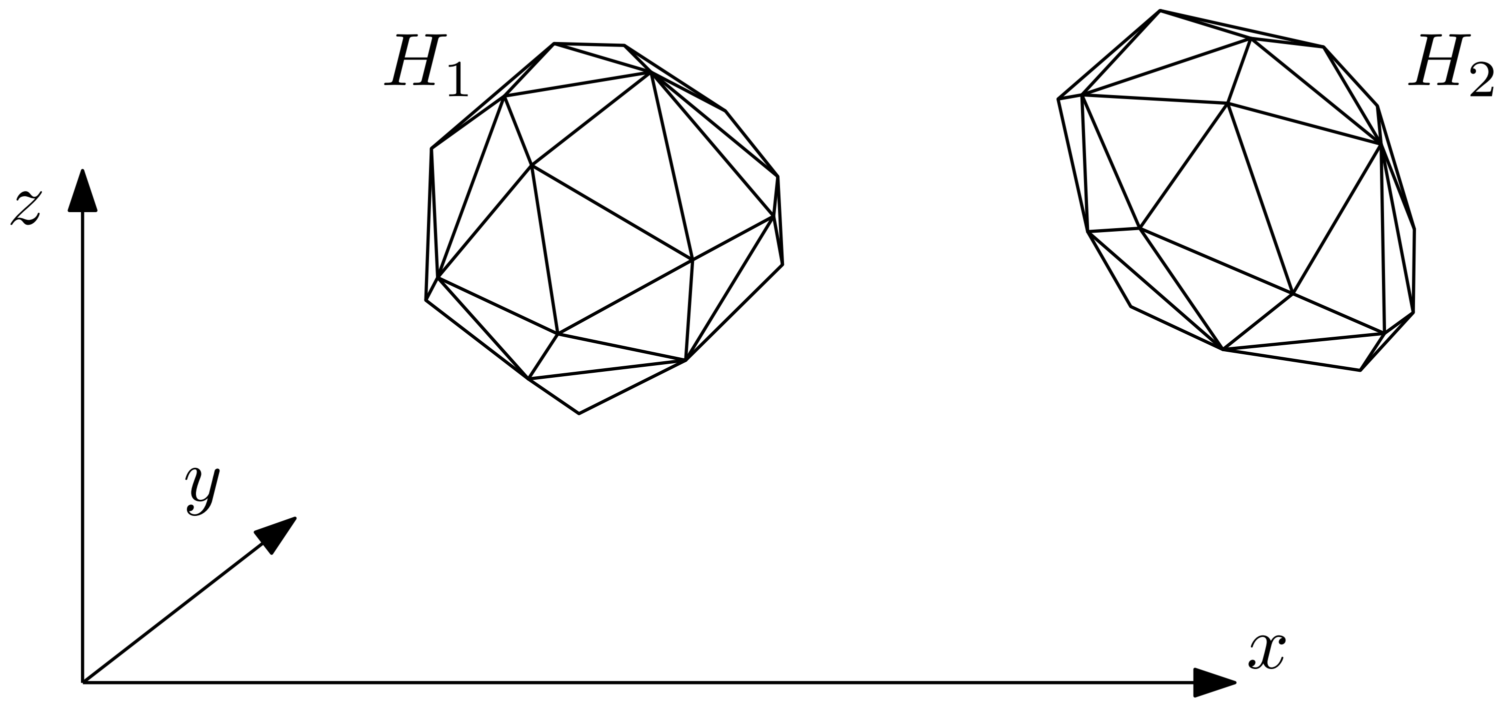
We need to merge in $O(n)$ time!

Starting the merge



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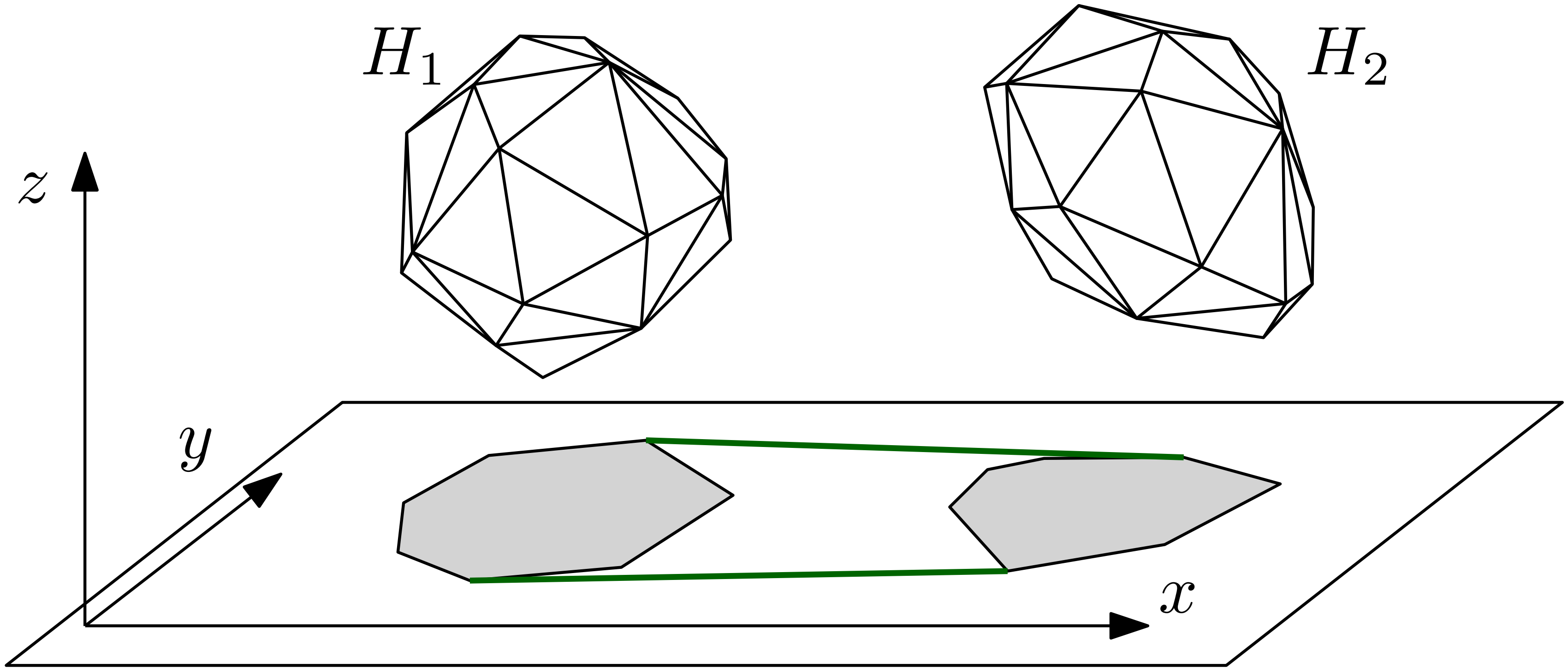

Project H_1 and H_2 to xy plane

Get common tangents (as in Assignment 1/7) $\rightarrow O(n)$

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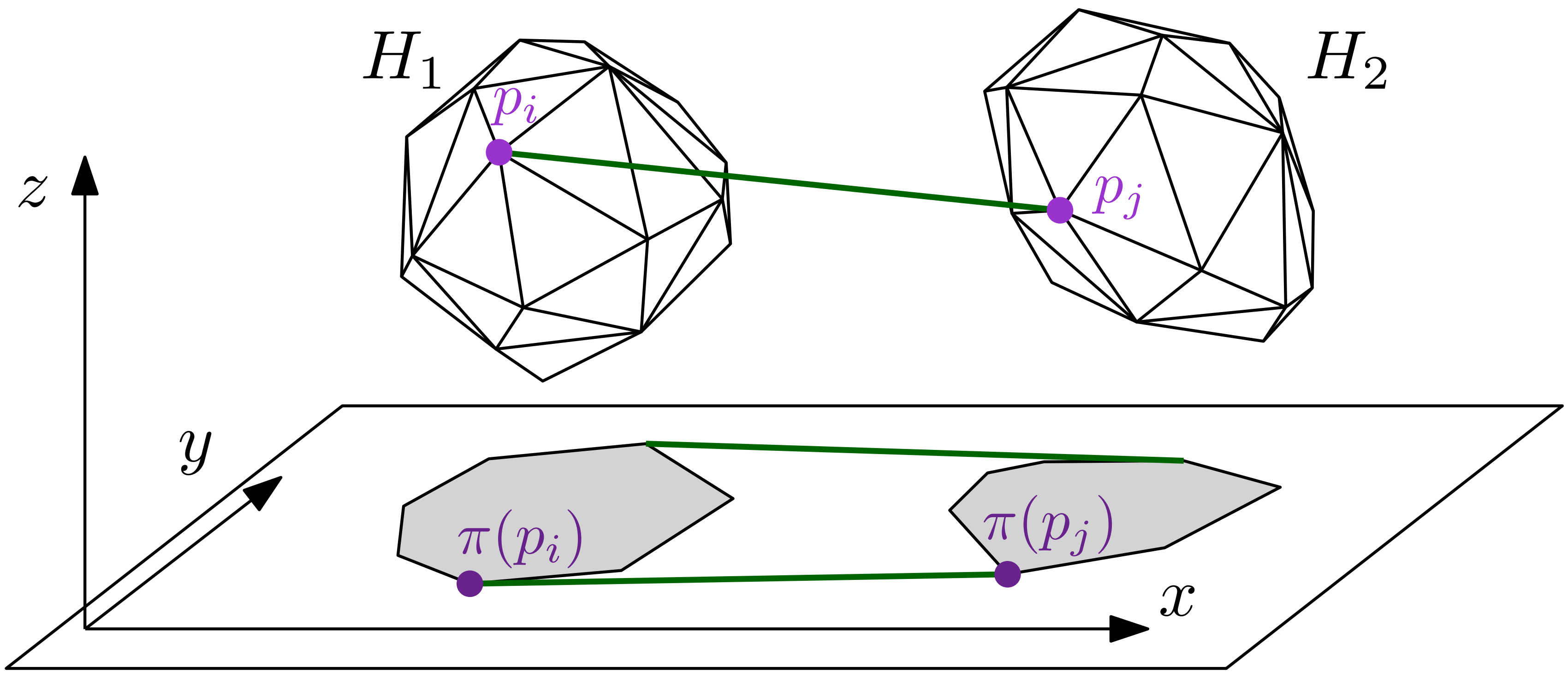

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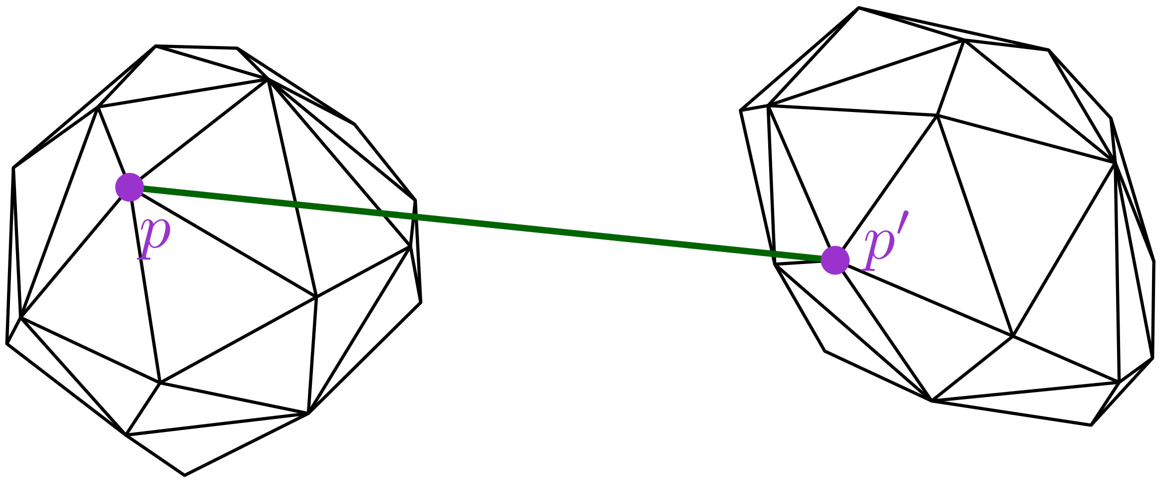
$\pi(p_i)\pi(p_j)$ is segment of $\pi(\mathrm{conv}(H_1)) \cup \pi(\mathrm{conv}(H_2))$

$\Rightarrow p_i p_j$ is a segment of $\mathrm{conv}(P)$.

Merging naively



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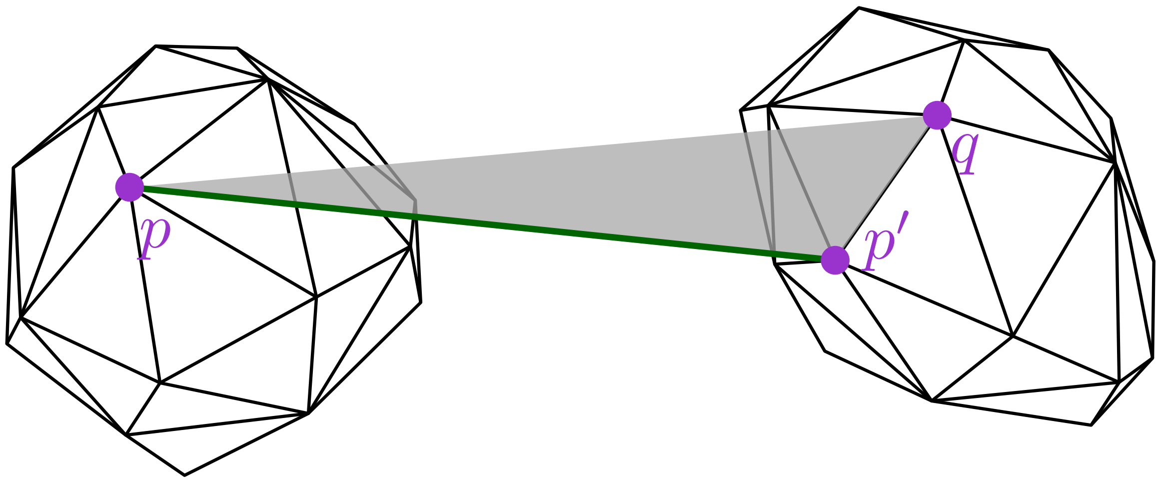

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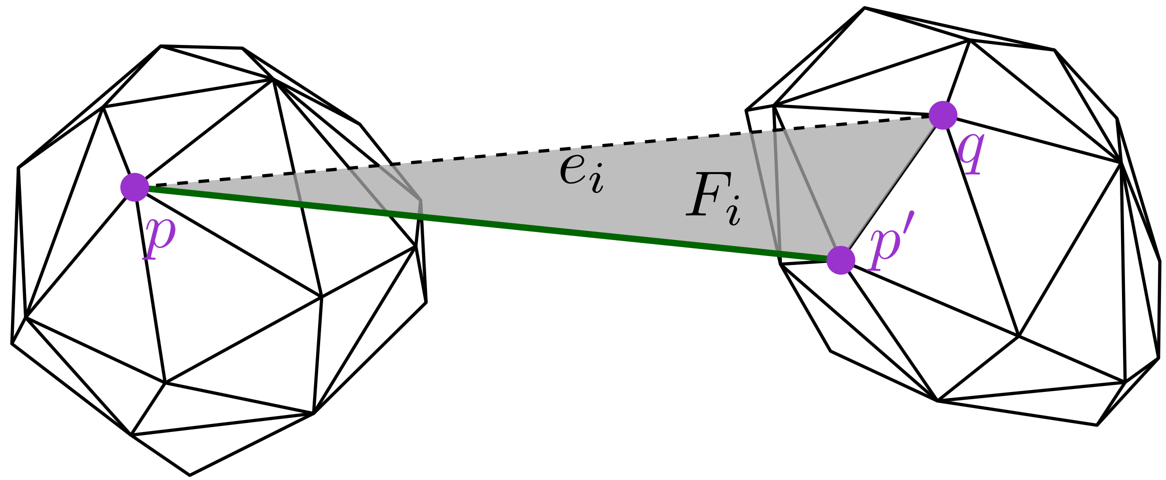

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Repeat: Find $p \in P$ maximizing $\sphericalangle\Big((\text{plane } e_i p), (\text{plane } F_i)\Big)$

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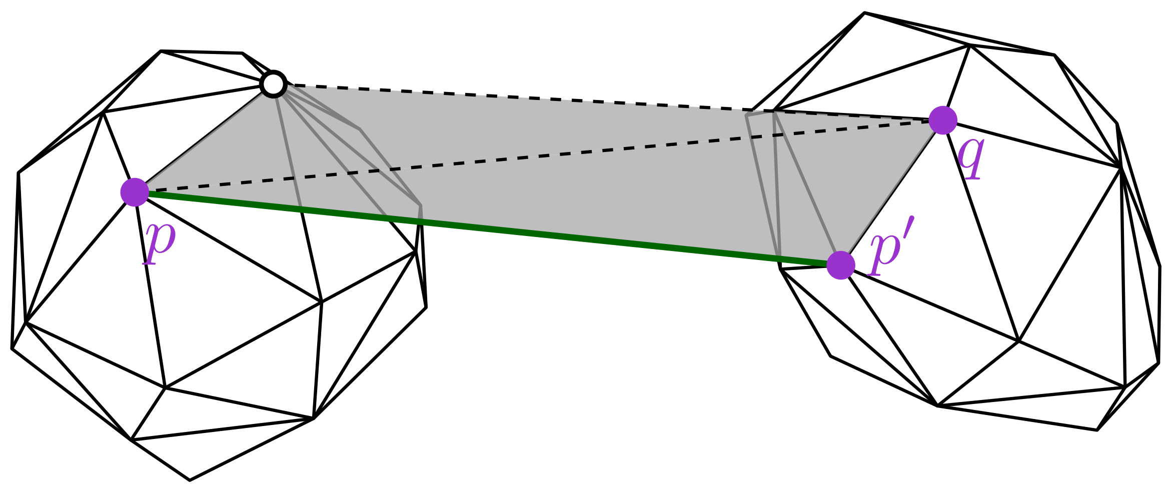

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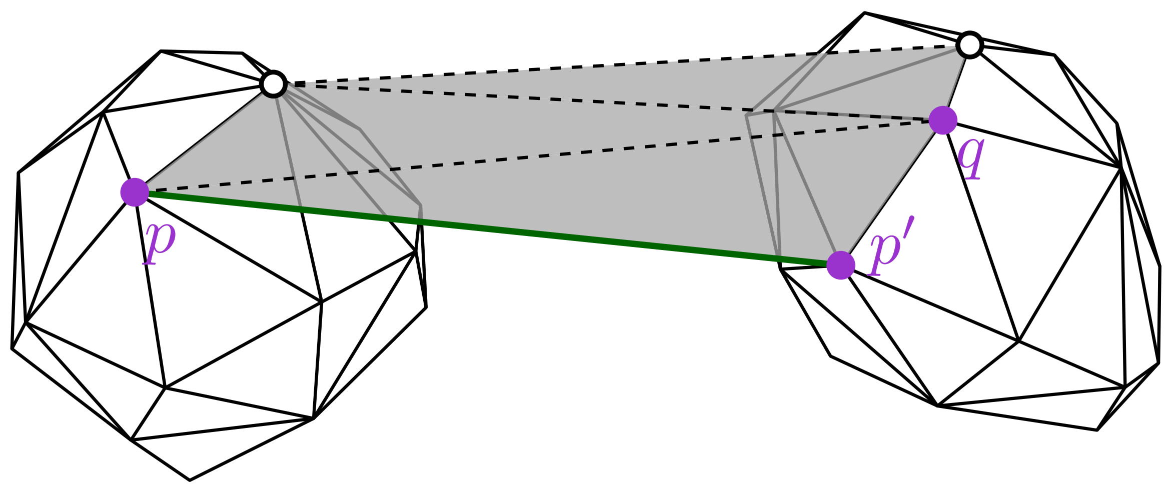

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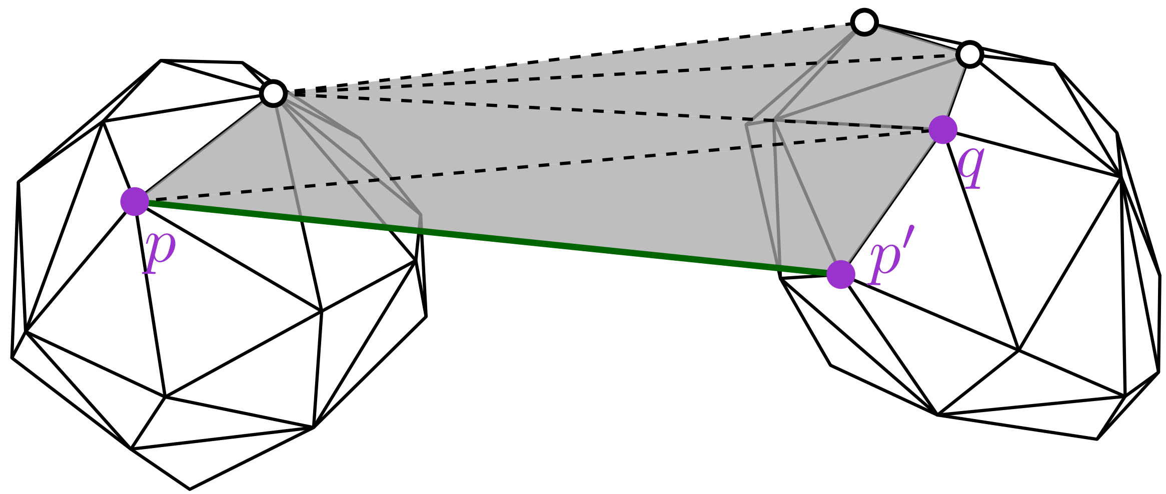

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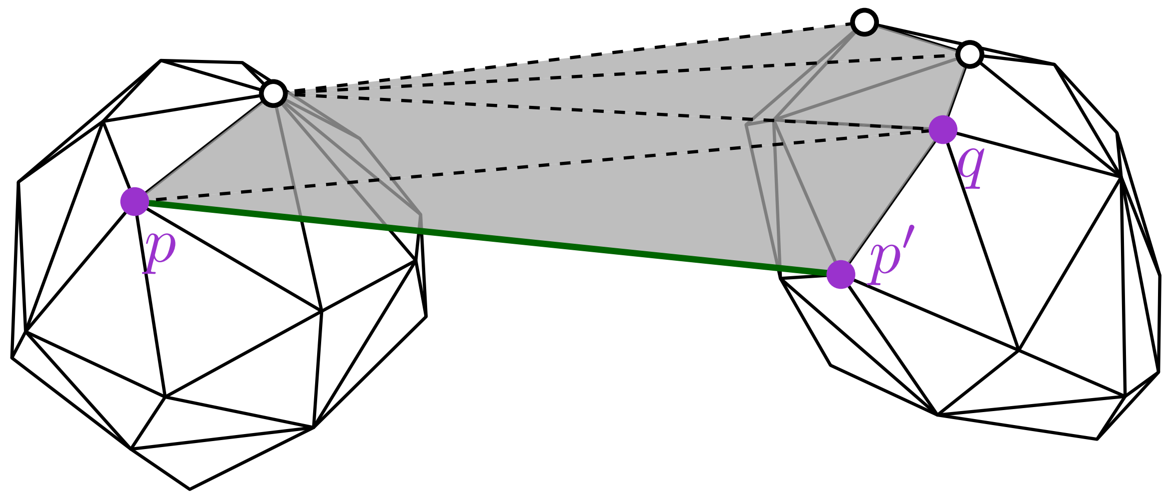

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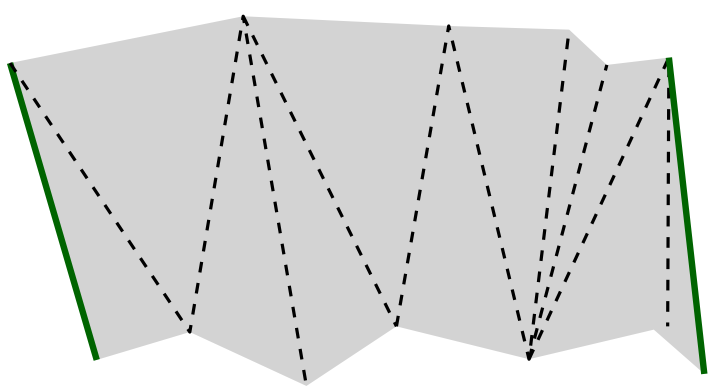
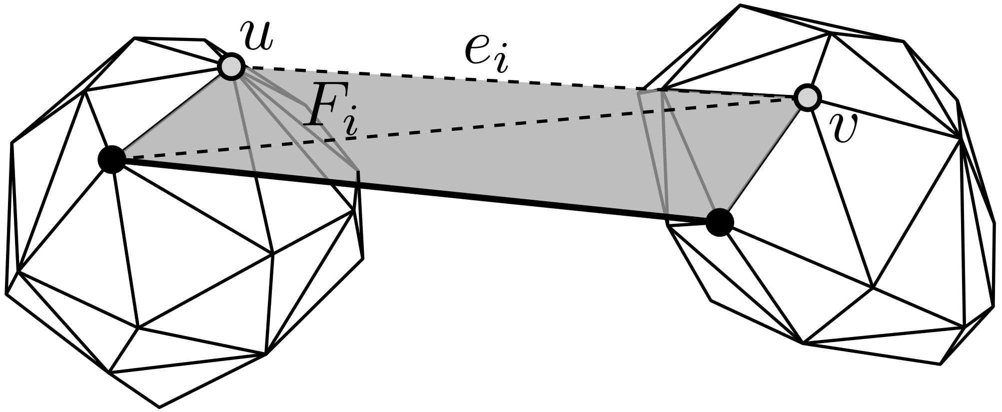


The result is a "cylinder".

Naive runtime: $O(n^2)$

Smart merge idea



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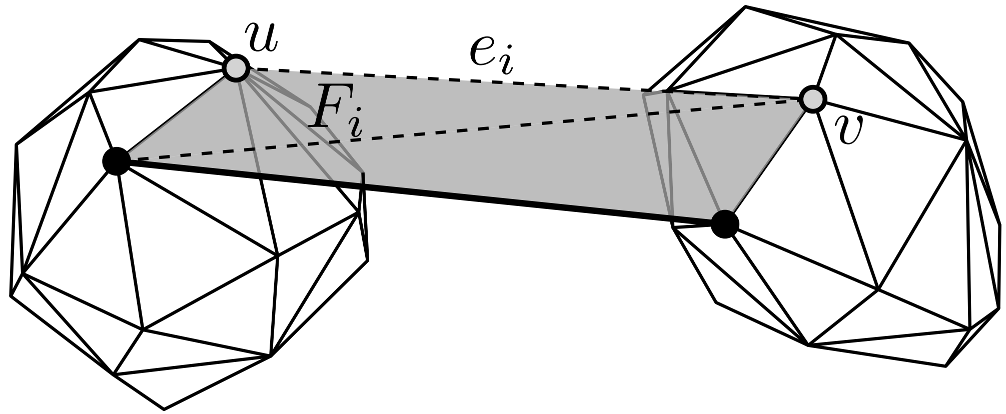

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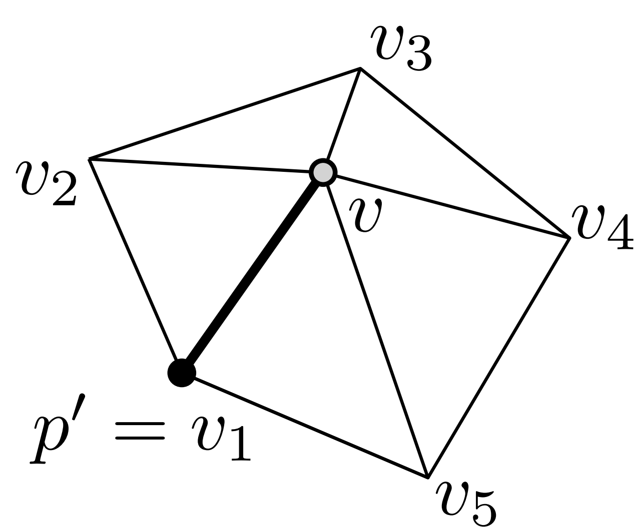

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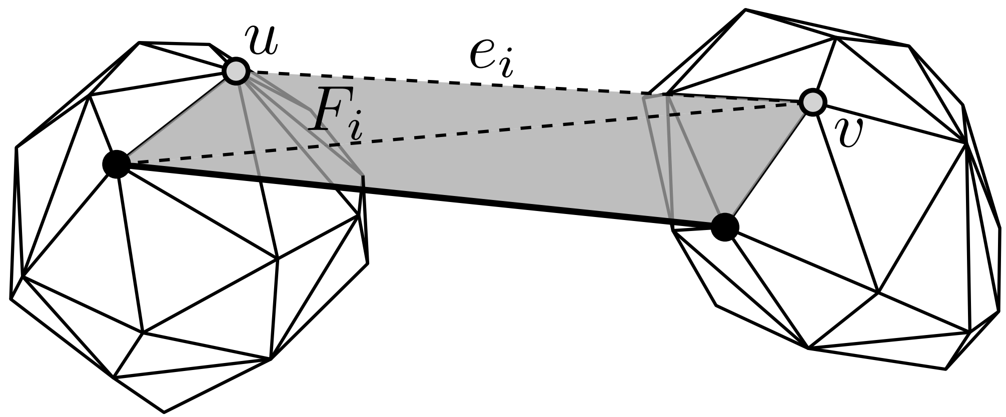

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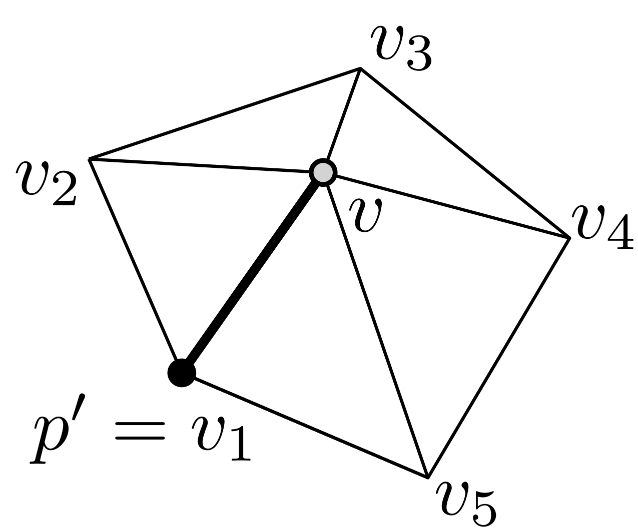

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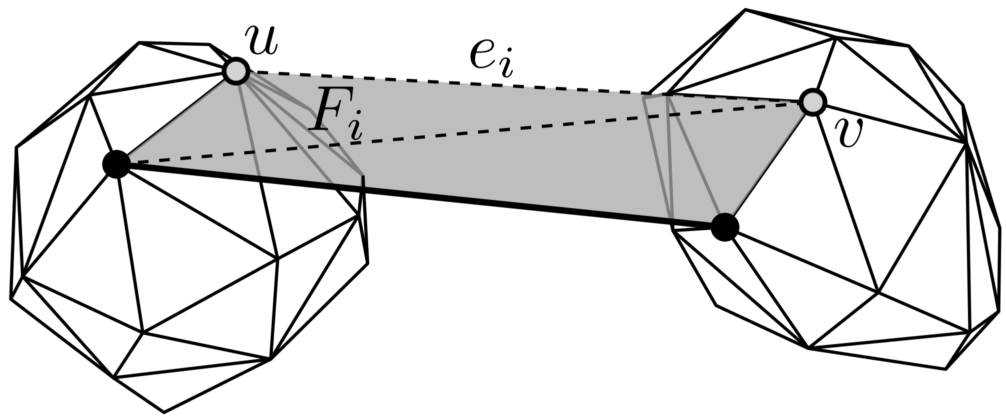

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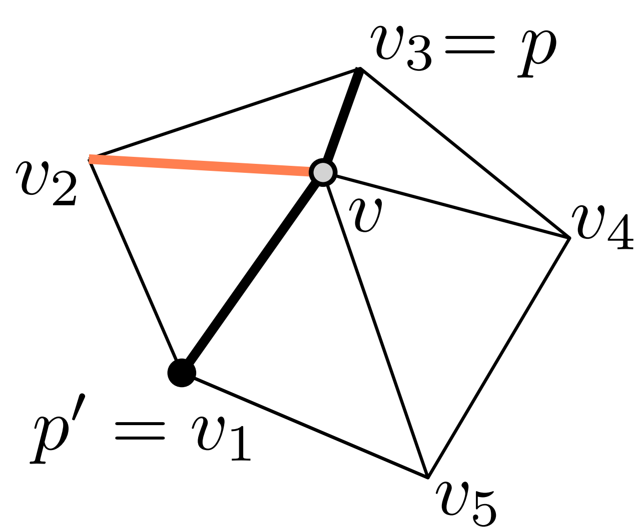
If $v_i \in p$ has largest angle
$\Rightarrow v_2, \ldots, v_{i-1}$ are *inside* $\mathrm{conv}(P)$!



Smart merge

function $\textsc{Merge}(H_1, H_2)$

  $uv =$ starting edge (form common tangent of $\pi(H_1)$ and $\pi(H_2)$)

  $\hat{u} = \text{argmax}_{u' \in N(u)} \sphericalangle\big(uvu', uv\pi(u)\big)$

  $\hat{v} = \text{argmax}_{v' \in N(v)} \sphericalangle\big(uvv', uv\pi(u)\big)$

  repeat

    if $\hat{v}$ has larger angle than $\hat{u}$ **then**

      $v_{prev} = v, \quad v = \hat{v}$

      Add face $F = uvv_{prev}$ to H

      If F is coplanar with previous face, merge.

      for $i = 2$ to $|N(v)|$ **do** ▷ in cw order

        if $\sphericalangle\big(uvv_i, uvv_{prev}\big) > \sphericalangle\big(uvv_{i-1}, uvv_{prev}\big)$ **then**

          Remove edge vv_{i-1} from H_2

        else

          $\hat{v} = v_{i-1}$, **Break**

    else

      (same, but swap v and u, H_2 and H_1, cw and ccw)

  until $uv =$ starting edge

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Face add: $O(1)$ (overall $O(n)$)

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Preparata–Hong divide&conquer takes $O(n \log n)$ time. ■

$\mathbb{R}^3$ Convex hull with rand. incremental construction (Clarkson and Shor 1989)

Ideas

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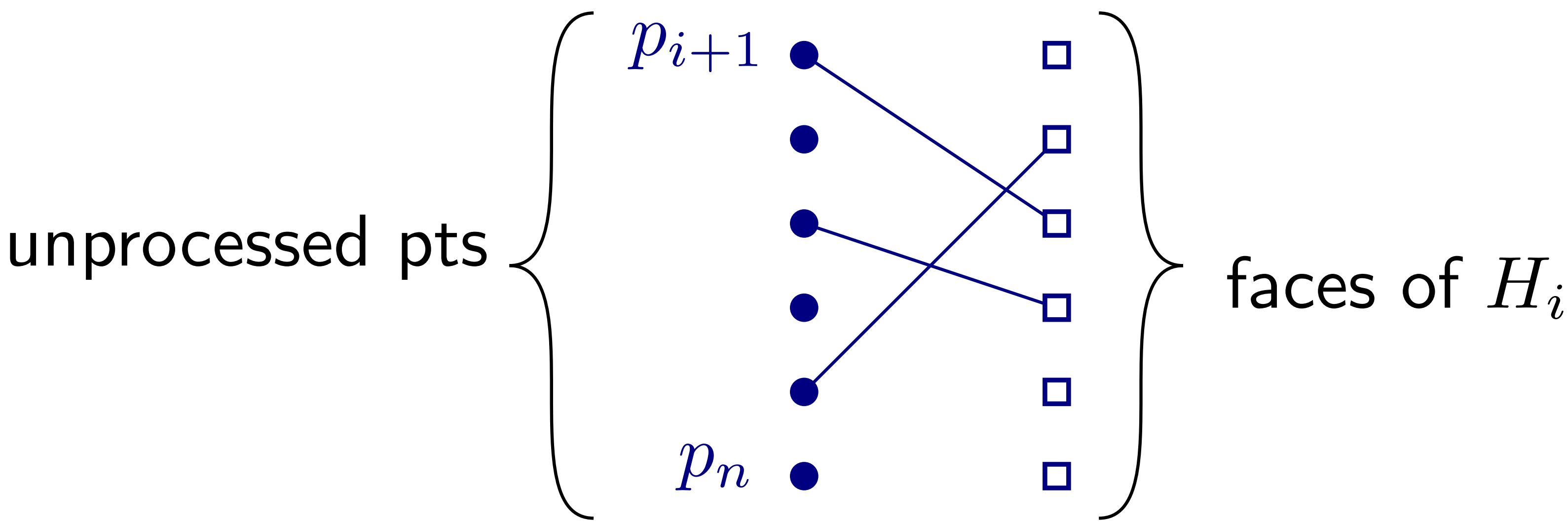

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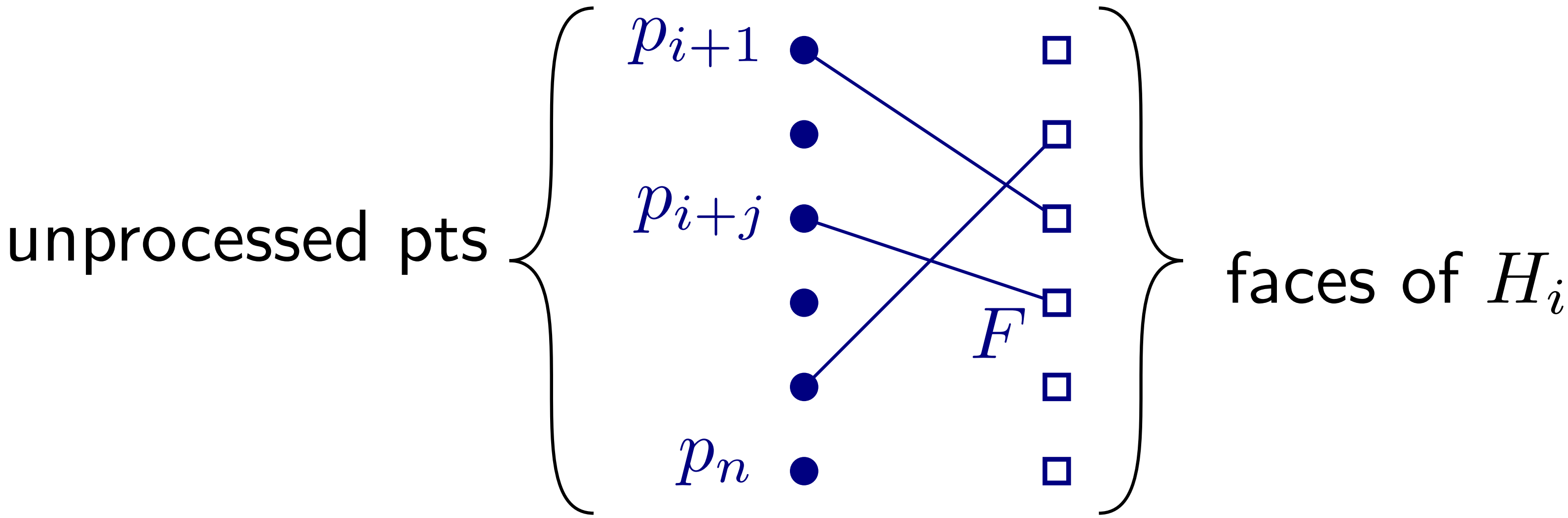

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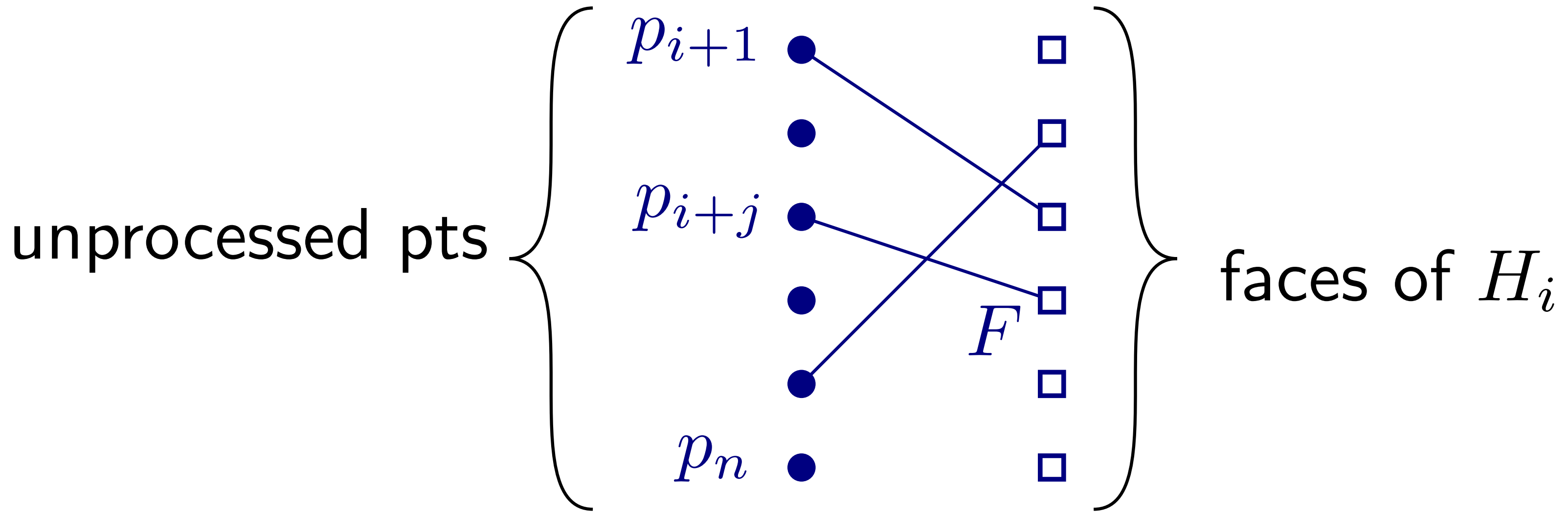

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$\Rightarrow$ if we add p_{i+j}, we must delete F

Algorithm overview

Find 4 points forming a tetrahedron, set up H_4

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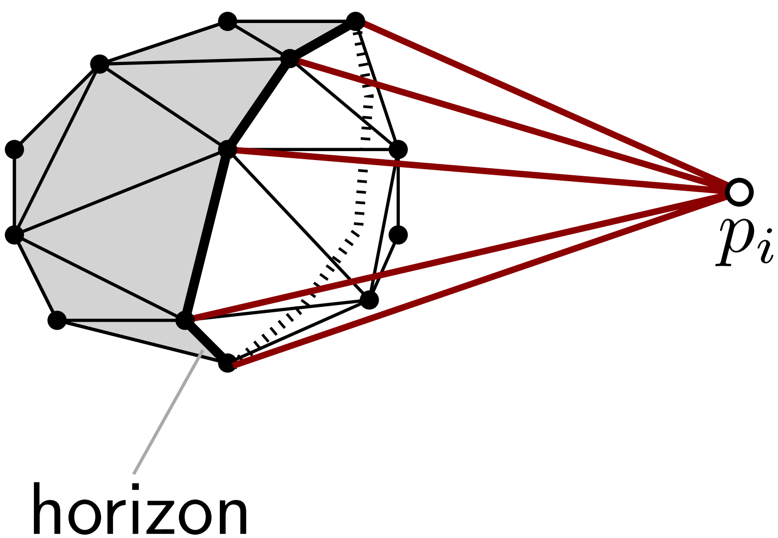

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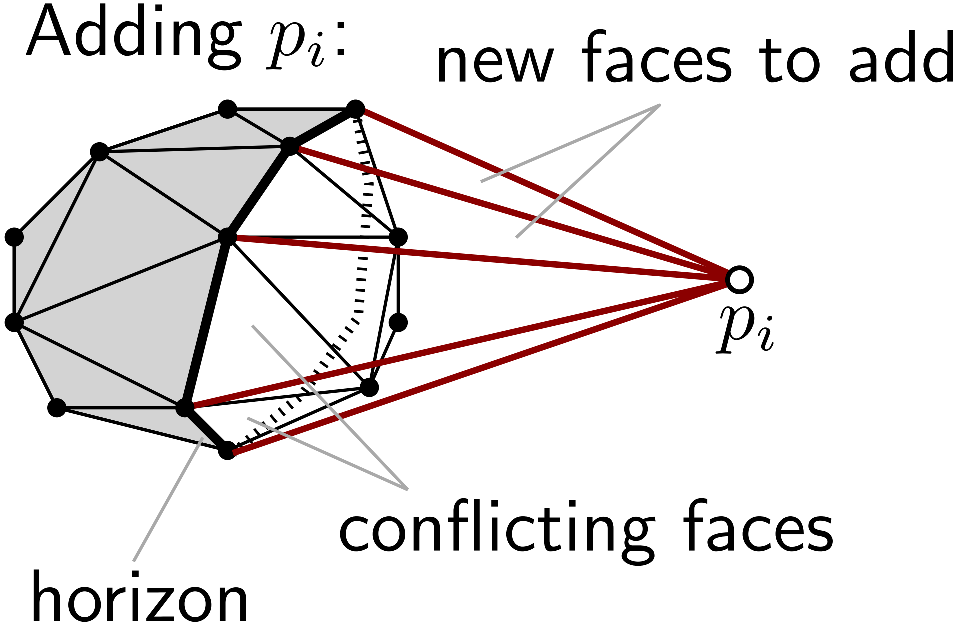

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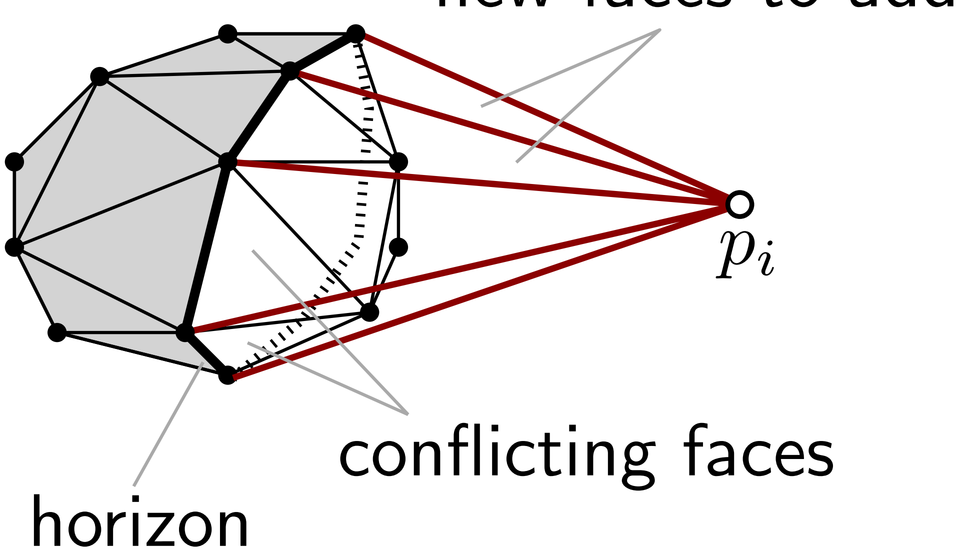

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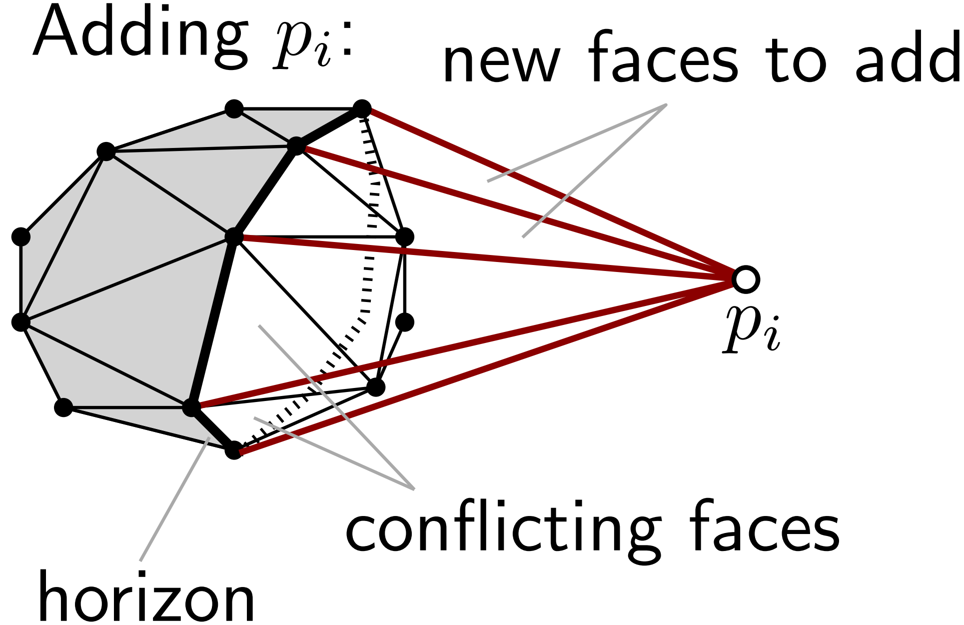

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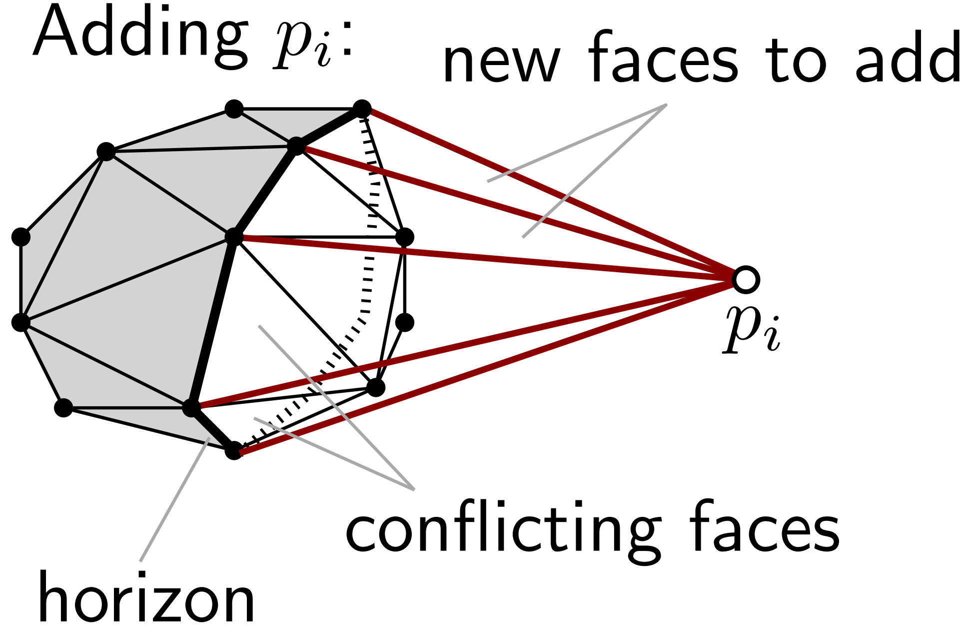

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if coplanar face:
$\rightarrow$ merge faces.
same conflict list!

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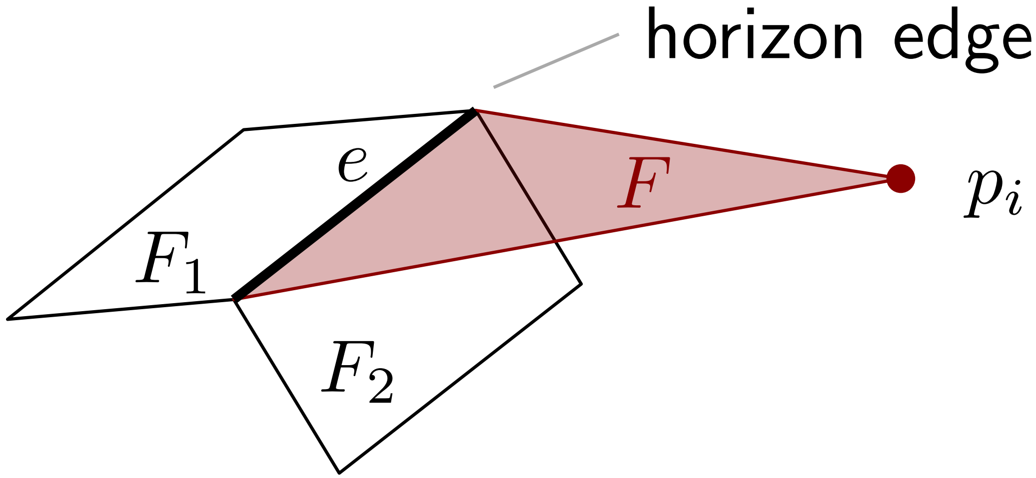

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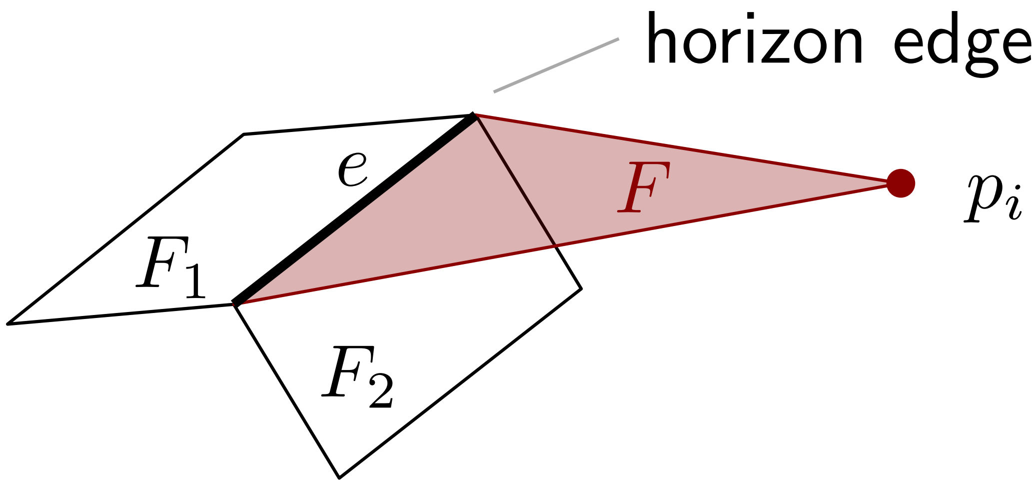

$$Conf(F) \subseteq \underbrace{Conf(F_1) \cup Conf(F_2)}$$

For each $p \subseteq U_i^e$ that sees F,
add conflict edge (F, p)

Running time analysis - lemmas

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Long proof, see Dutch book

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$$\Rightarrow \mathbb{E}(\text{total \#created faces}) = 4 + \sum_{j=5}^{n} \mathbb{E}(\deg^j(p_j)) \leq 6n-20.$$

□

Chan's algorithm in $\mathbb{R}^3$

Chan's algorithm recap and changes

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Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

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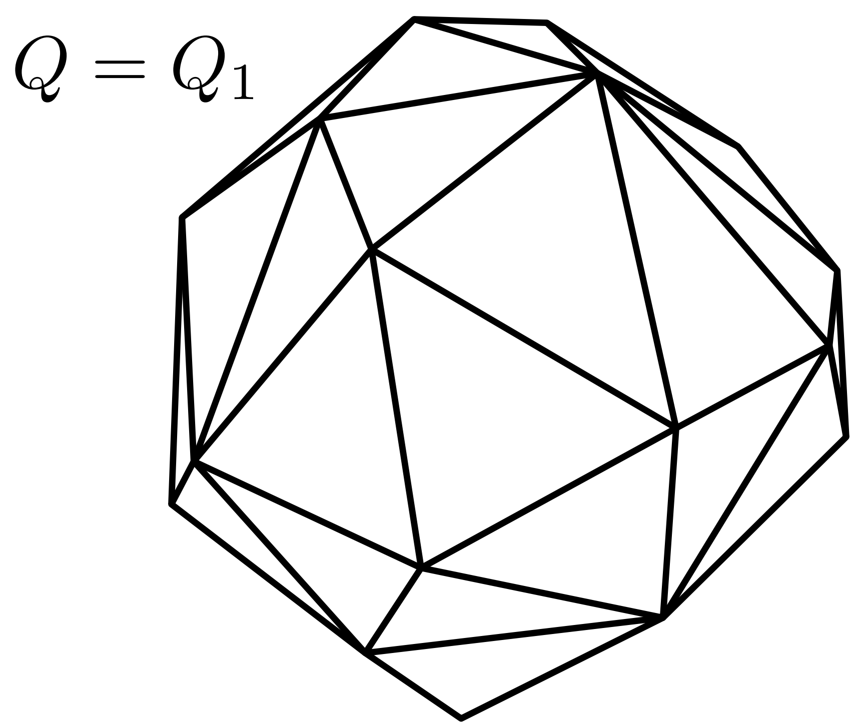

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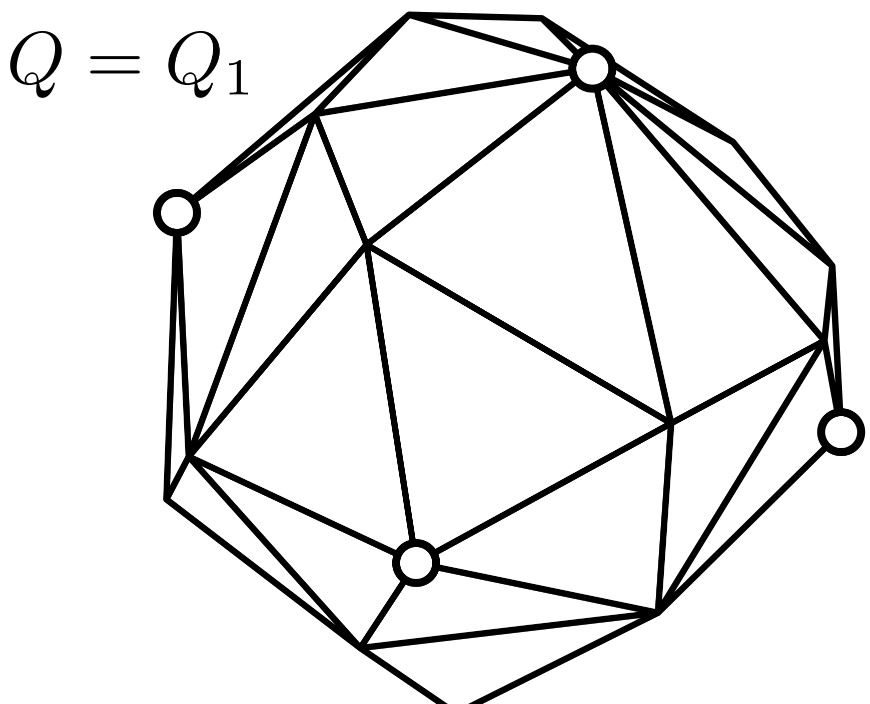

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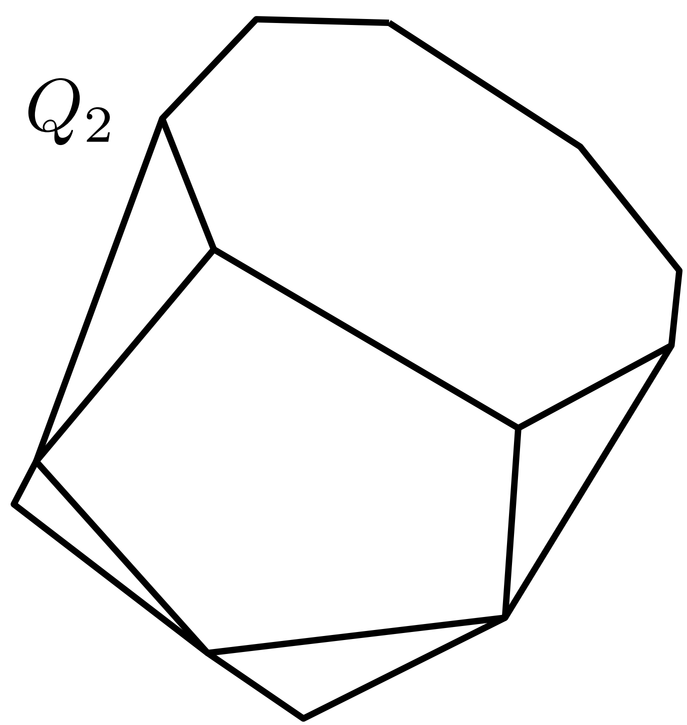

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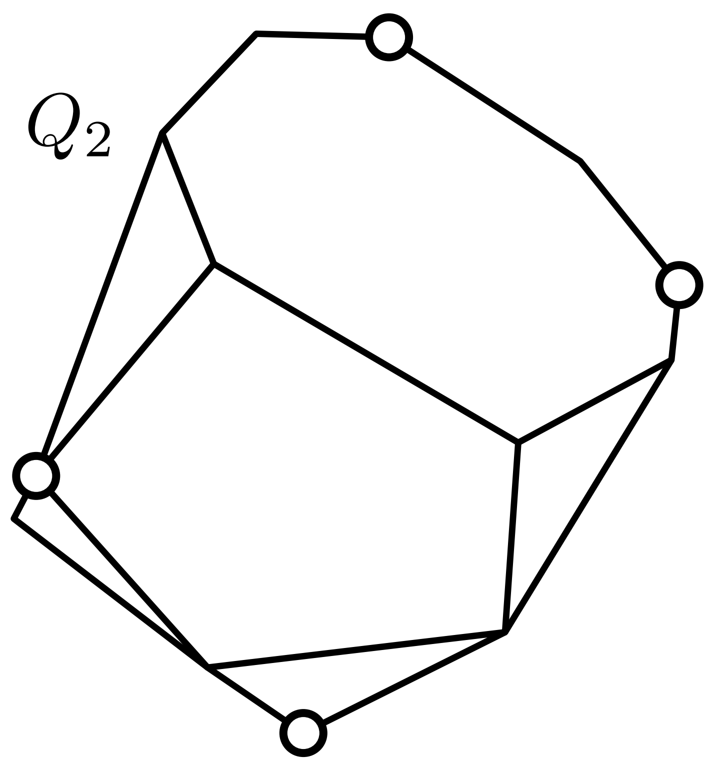

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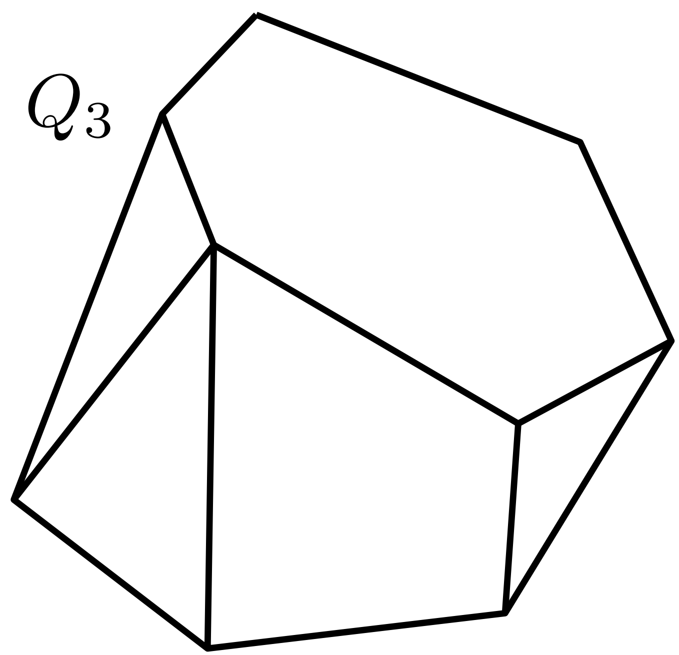

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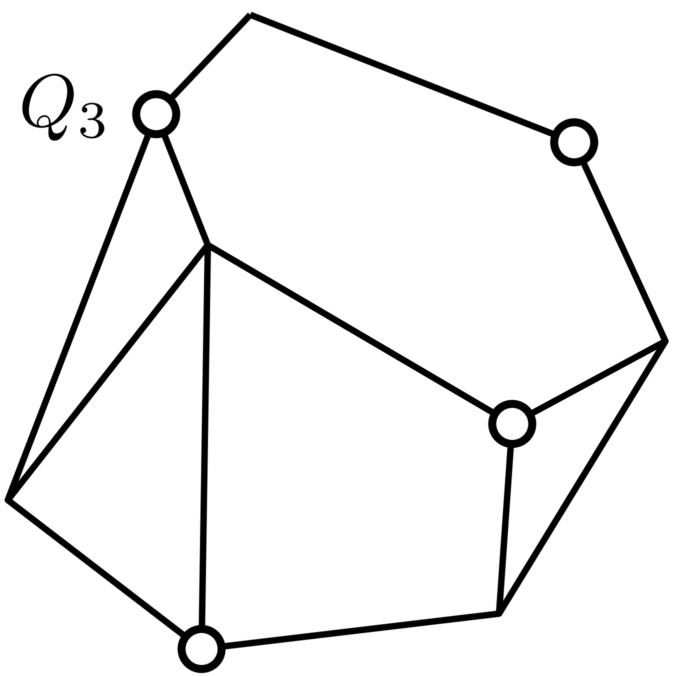

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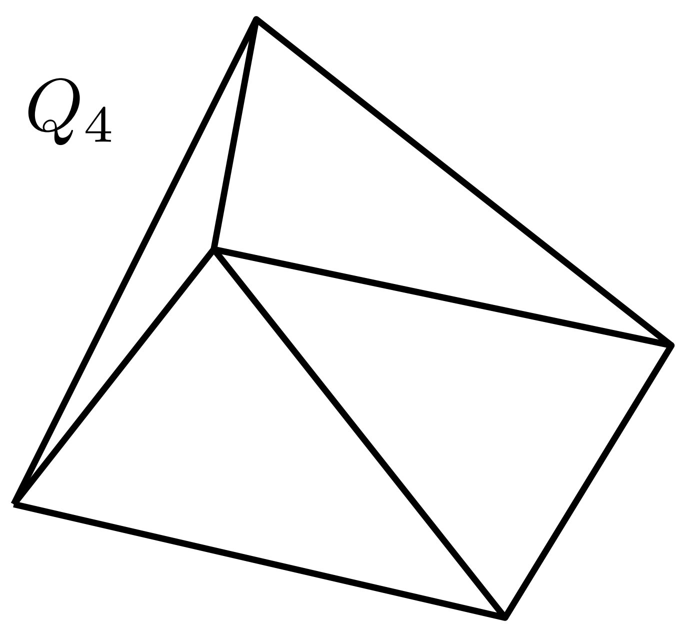

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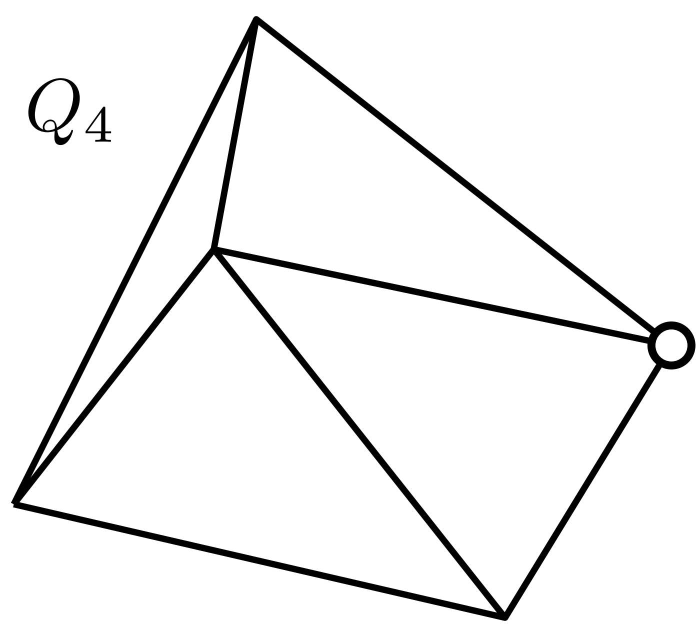

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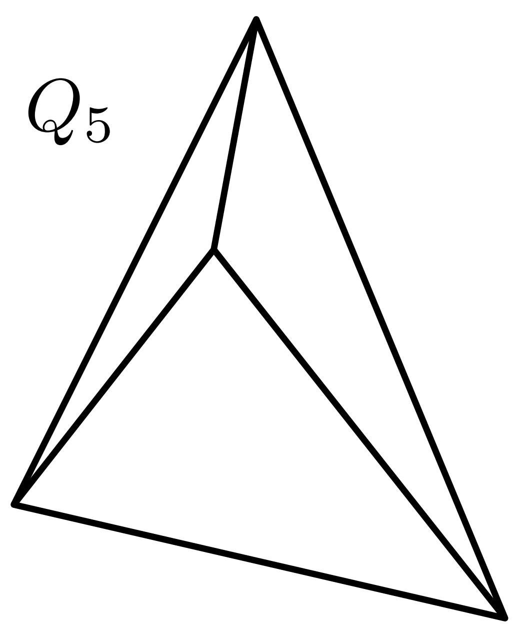

Constructing the DK hierarchy

Theorem Given Q, a DK hierarchy with $k = O(\log n)$, size $\sum_{i=1}^{k}(|V(Q_i)|) = O(n)$ and degree

$$\max_i \max\{\deg_{G(Q_i)}(v) \mid v \in V(Q_i) \setminus V(Q_{i+1})\} \leq 11$$

can be computed in $O(n)$ time.

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Proof. Iteratively remove set S, a greedy maximal independent set among vertices of degree ≤ 11.

Claim: $|S| \geq |V(Q)|/24$.
Suppose not: $|S| < |V(Q)|/24$
$\Rightarrow \bigcup_{s \in S} N[s] < |V(Q)|/2$
$\Rightarrow G(Q)$ has $\geq |V(Q)|/2$ vertices of degree ≥ 12
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Euler's formula: $E(Q) \leq 3|V(Q)| - 6$

□

for $i = 1$ to $\lceil \log\log(3n-6) \rceil$ **do**
  $m = 2^{2^i}$
  for $j = 1$ to $\lceil n/m \rceil$ **do**
    Create group P_j
    $H_j =$Preparata–Hong(P_j)
    Compute dual D–K hierarchy of H_j
  $F_0 =$ starting face, e_1, e_2, e_3: ccw arcs of $E(F)$,
  $H.add(F_0)$, $Queue = (e_1,e_2,e_3)$, $Seen = \{e_1,e_2,e_3\}$,
  for $s = 2$ to m **do**
    $e = Queue.next$
    for $j = 1$ to $\lceil n/m \rceil$ **do**
      $q^j = TangentFind(e, dualDK(H_j))$
    $q =$ point q^j maximizing $\sphericalangle\big((\text{plane } e, q^j), F(e)\big)$
    $H.add$(face e, q), $e', e'' =$ next arcs of face e, q
    if $e' \notin Seen$ **then** $Queue.add(e'), Seen.add(e')$
    if $e'' \notin Seen$ **then** $Queue.add(e''), Seen.add(e'')$
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Self-balancing BST

max size$= h$

    $e = Queue.next$

    for $j = 1$ to $\lceil n/m \rceil$ **do**

      $q^j = TangentFind(e, dualDK(H_j))$

    $q =$ point q^j maximizing $\sphericalangle\big((\text{plane } e, q^j), F(e)\big)$

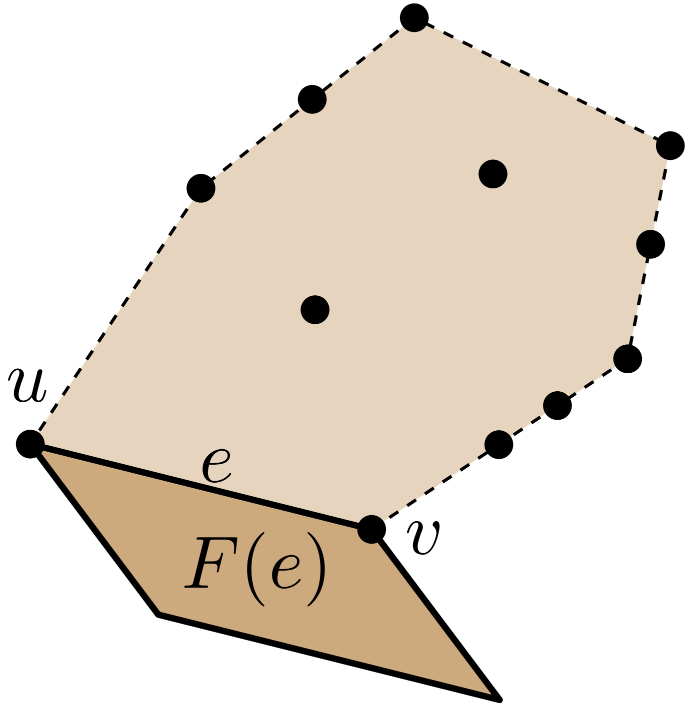
    $H.add$(face e, q), $e', e'' =$ next arcs of face e, q

    if $e' \notin Seen$ **then** $Queue.add(e'), Seen.add(e')$

    if $e'' \notin Seen$ **then** $Queue.add(e''), Seen.add(e'')$

return H

Handling degeneracies

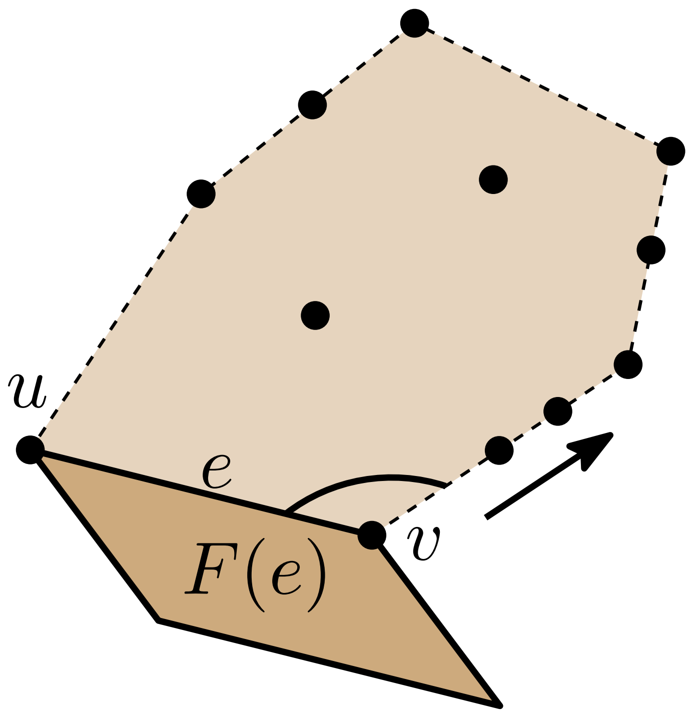


Handling degeneracies

Among $Q = \operatorname{argmax}_{q'} \left(\sphericalangle F(e), (\text{plane } uvq')\right)$

q should maximze $\sphericalangle(uvq)$

and among these maximize $\operatorname{dist}(v, q)$



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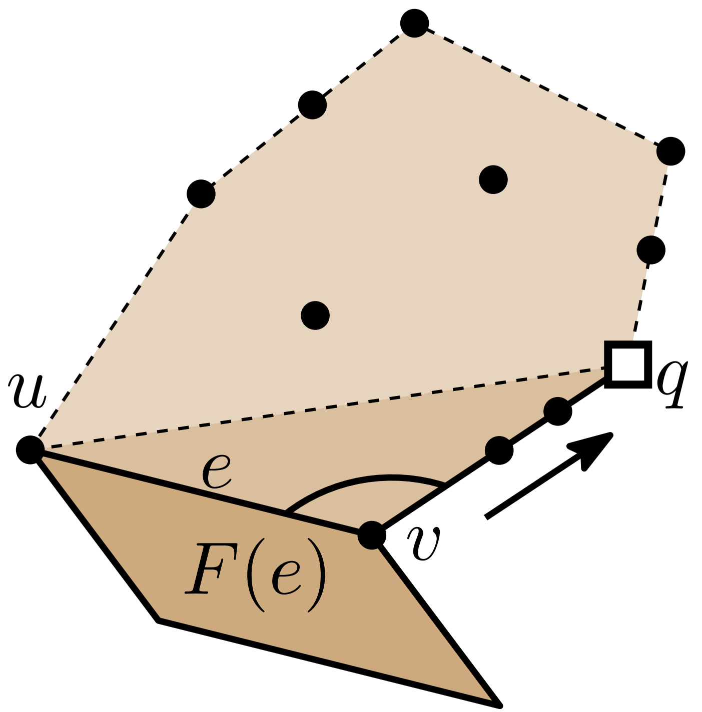

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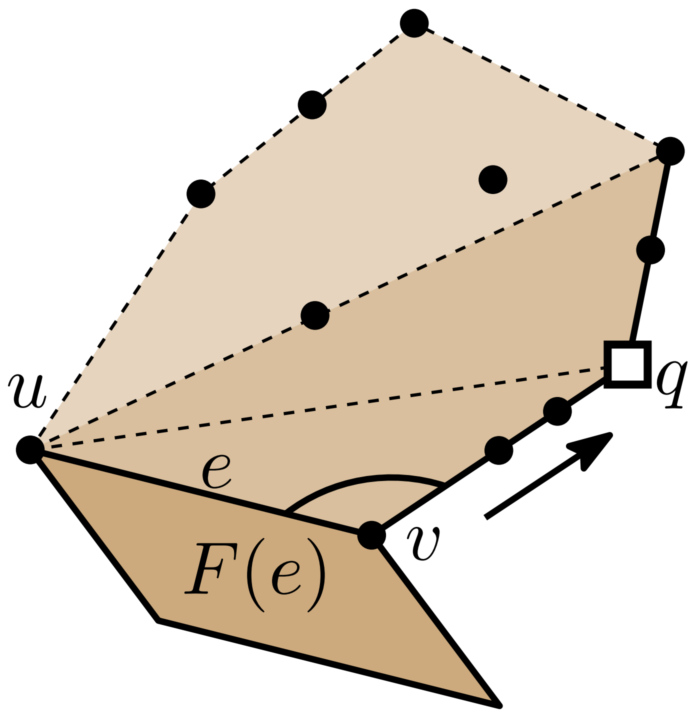

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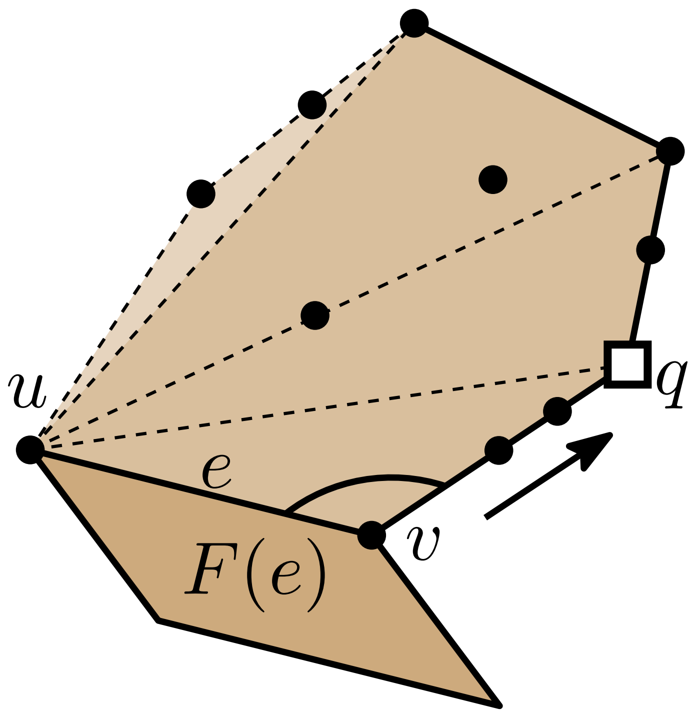

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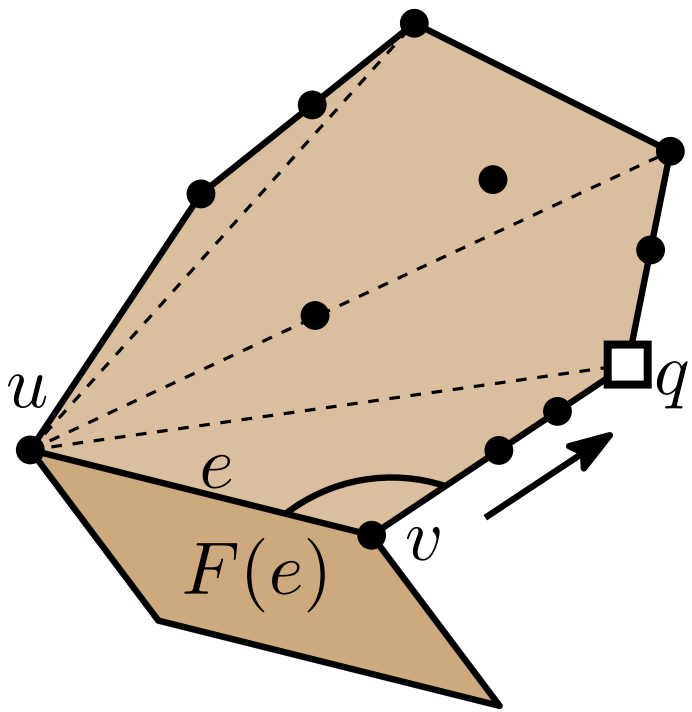

H is a triangulation of the hull.

Postprocess: merge at edge if neighboring faces are coplanar

$(\rightarrow O(n))$

Higher-dimensional convex hulls

Many faces, many facets

In $\mathbb{R}^d$, we have $0, 1, \ldots, (d-1)$-dimensional faces.

vertices   edges   facets

Many faces, many facets

In $\mathbb{R}^d$, we have $0, 1, \ldots, (d-1)$-dimensional faces.

vertices edges facets

Given a set P of n points in $\mathbb{R}^d$, compute:

(a) all facets of $\mathrm{conv}(P)$

(b) all vertices of $\mathrm{conv}(P)$

(c) all faces of $\mathrm{conv}(P)$

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There can be up to $\Theta(n^{\lfloor d/2 \rfloor})$ facets!

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Let P be distinct points on the moment curve

$$\{(x, x^2, x^3, \ldots, x^d) \mid x \in \mathbb{R}\}.$$

Then $\mathrm{conv}(P)$ has the maximum number of k-faces for all $k \in [d]$.

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$O(n \log n + n^{\lfloor d/2 \rfloor})$ achieved by many algorithms