Geometric Spanners

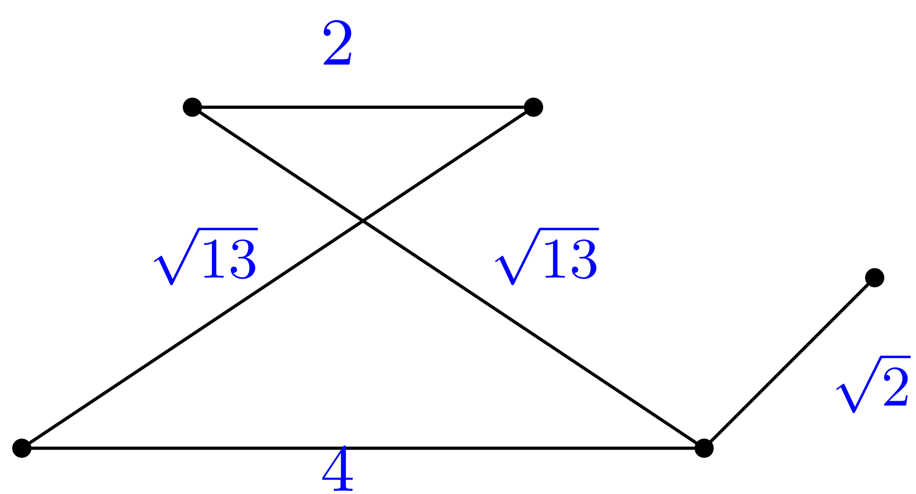
S finite set of points in $\mathbb{R}^d$

Geometric graph on S*:* edge-weighted graph

vertex set S,

edges correspond to straight segments connecting points in S

weight of an edge is its euclidean length



Geometric Spanners

S finite set in $\mathbb{R}^d$ "stretch factor" $t > 1$

t-spanner for S:
a geometric graph G on S so that for every $p, q \in S$ you have $d_G(p, q) \leq t \cdot \delta(p, q)$

$d_G()$ ... shortest path distance in G
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Goal: Given S and $t > 1$
prove existence/find t-spanner G for S s.t.

- G has few edges ($O(n)$)
- G is planar
- G has small maximum degree ($O(1)$)
- G has small total edge weight ($O(wt(MST(S))$)

- Construction takes little time

Delaunay triangulations as spanners

Theorem: (Dobkin, Friedman, Supowit)
For S in the plane the Delaunay triangulation of S is a t-spanner for S with $t \leq (1 + \sqrt{5})\pi/2 \approx 5.08$

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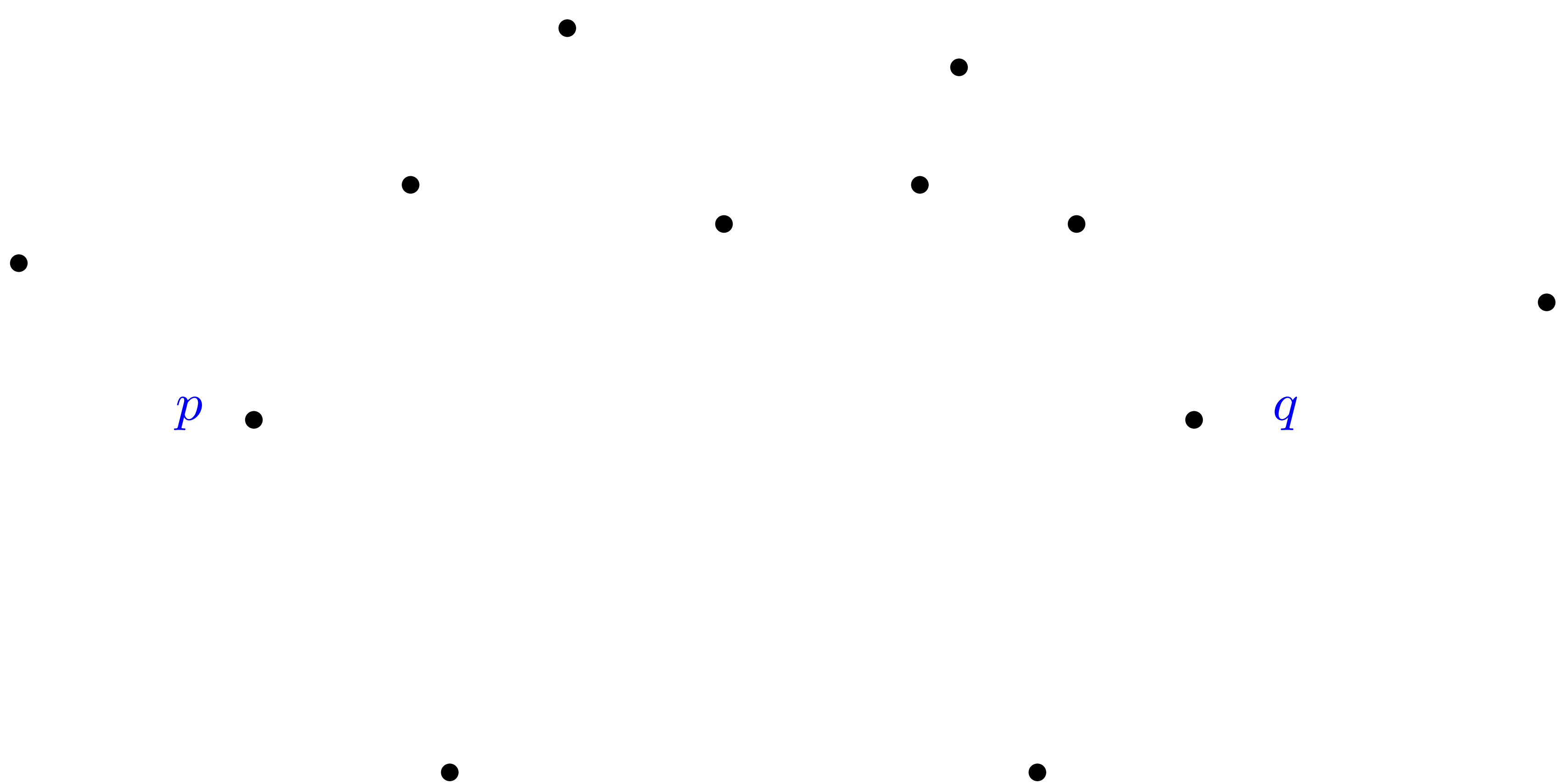

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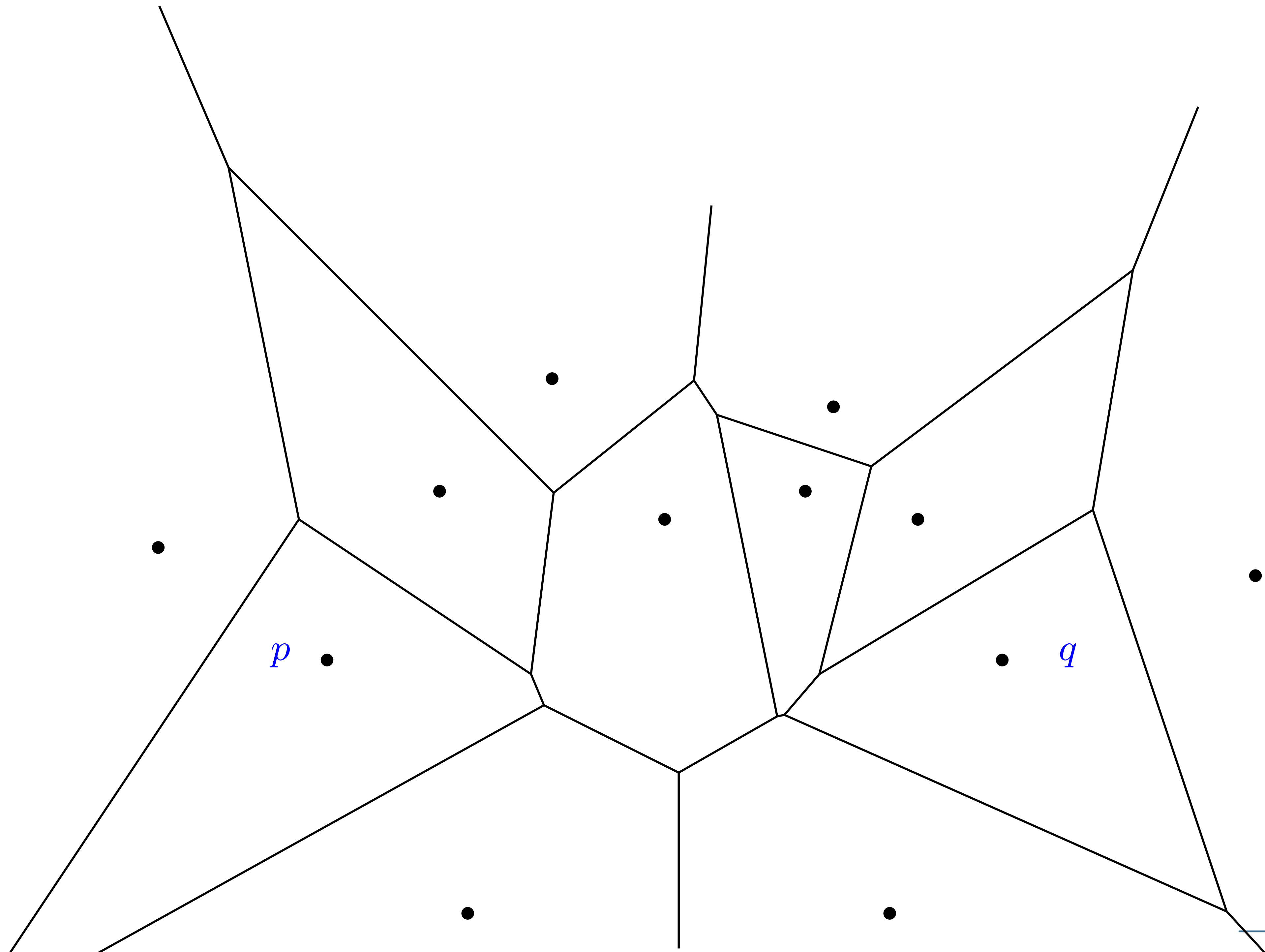

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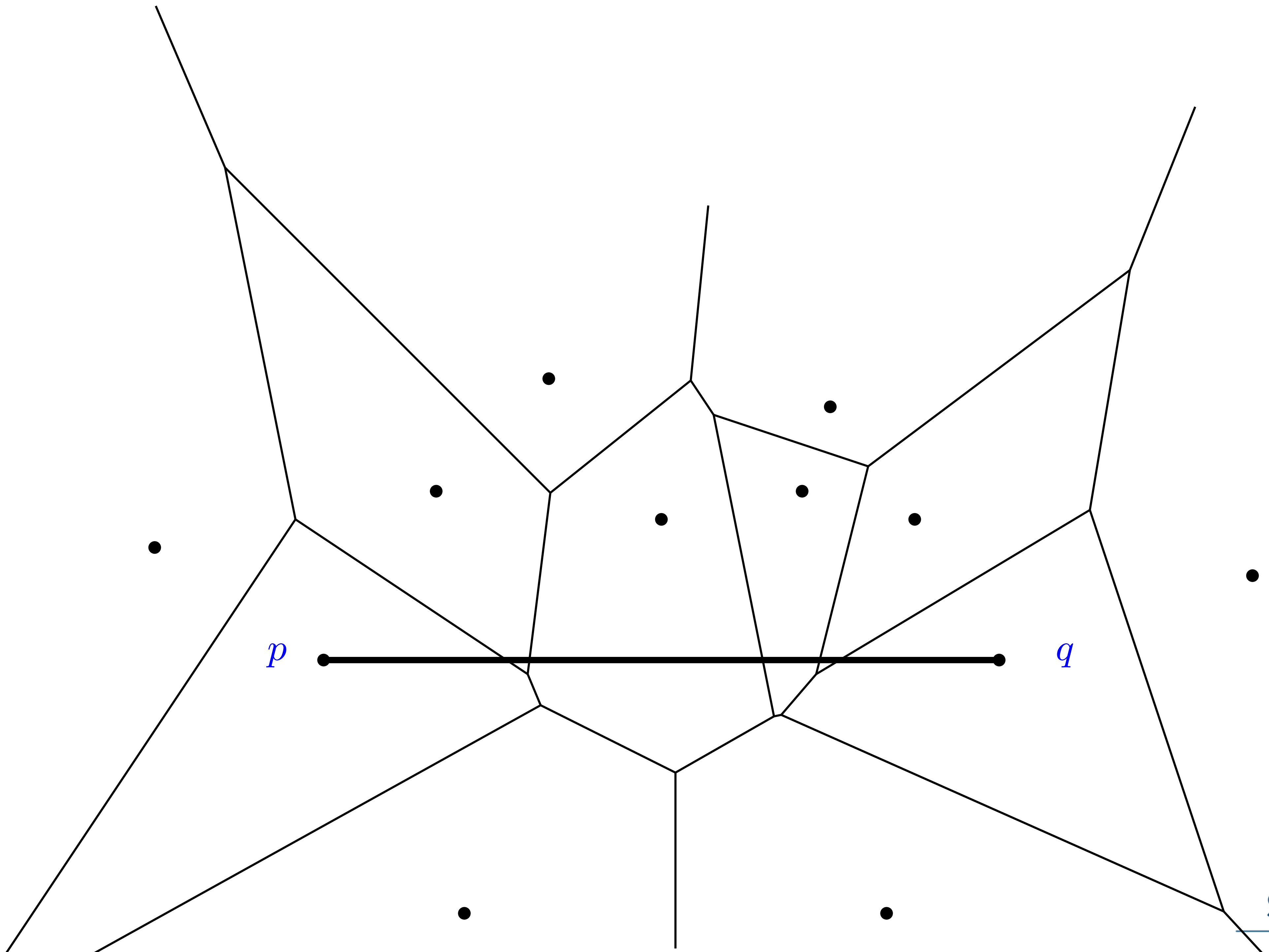

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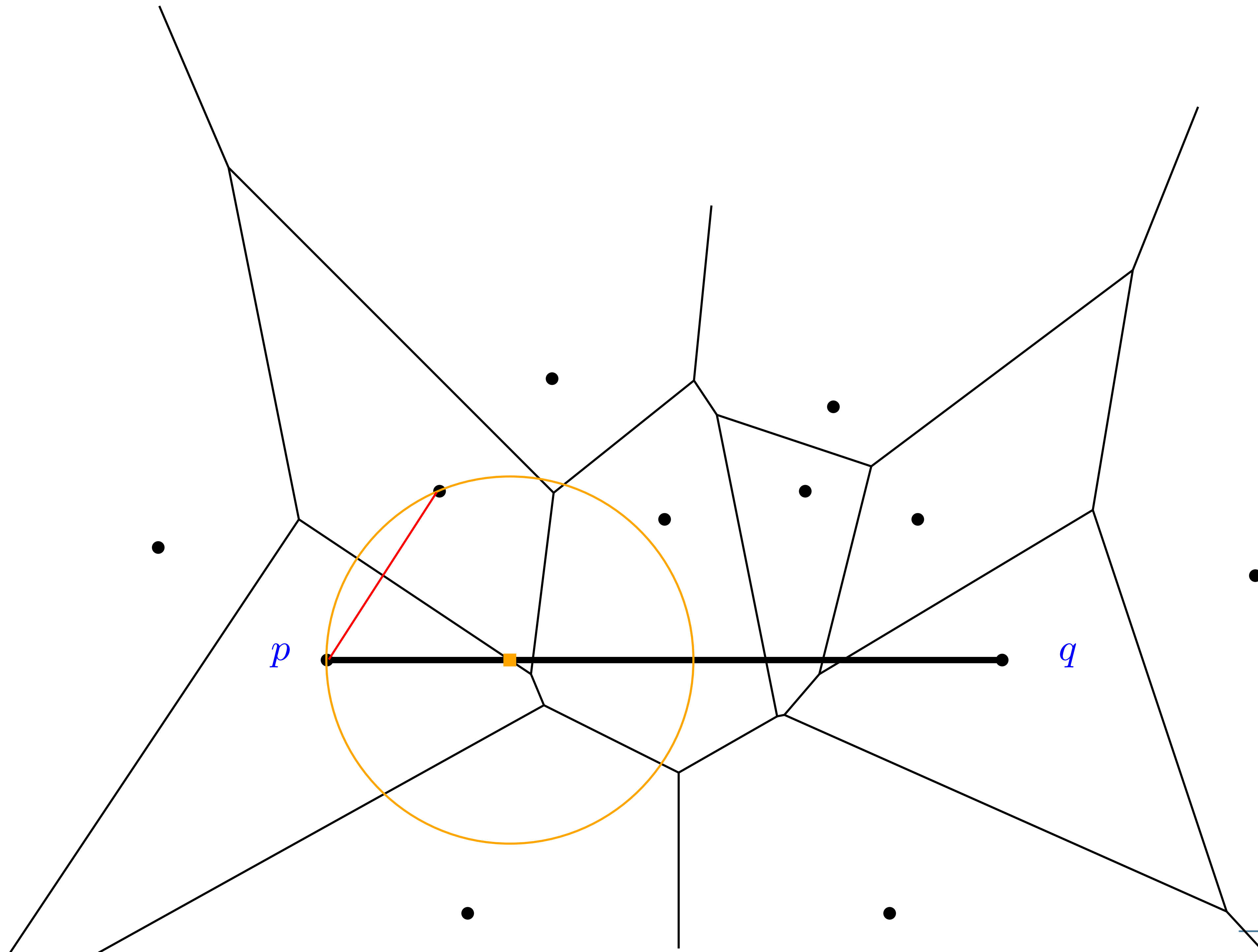

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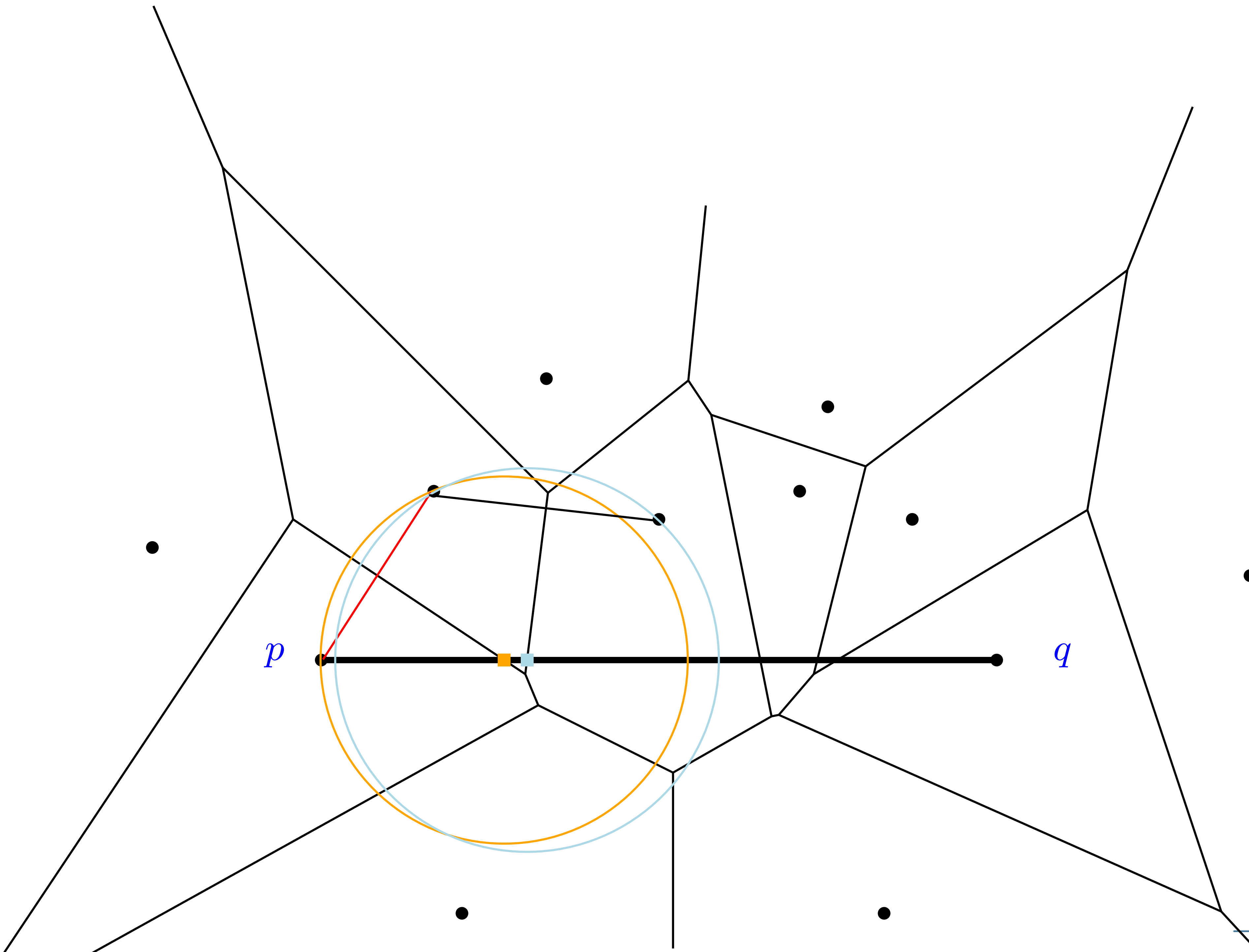

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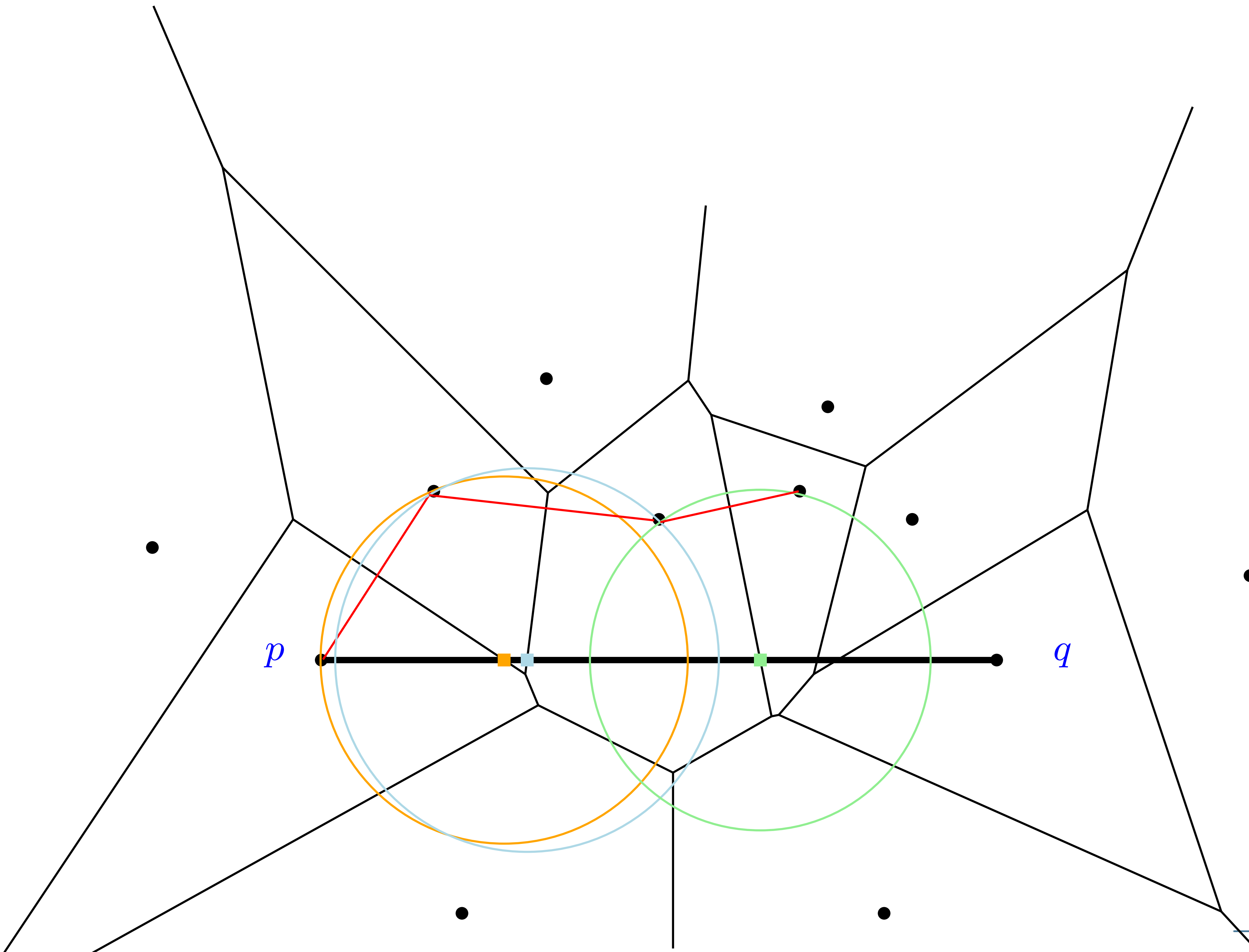

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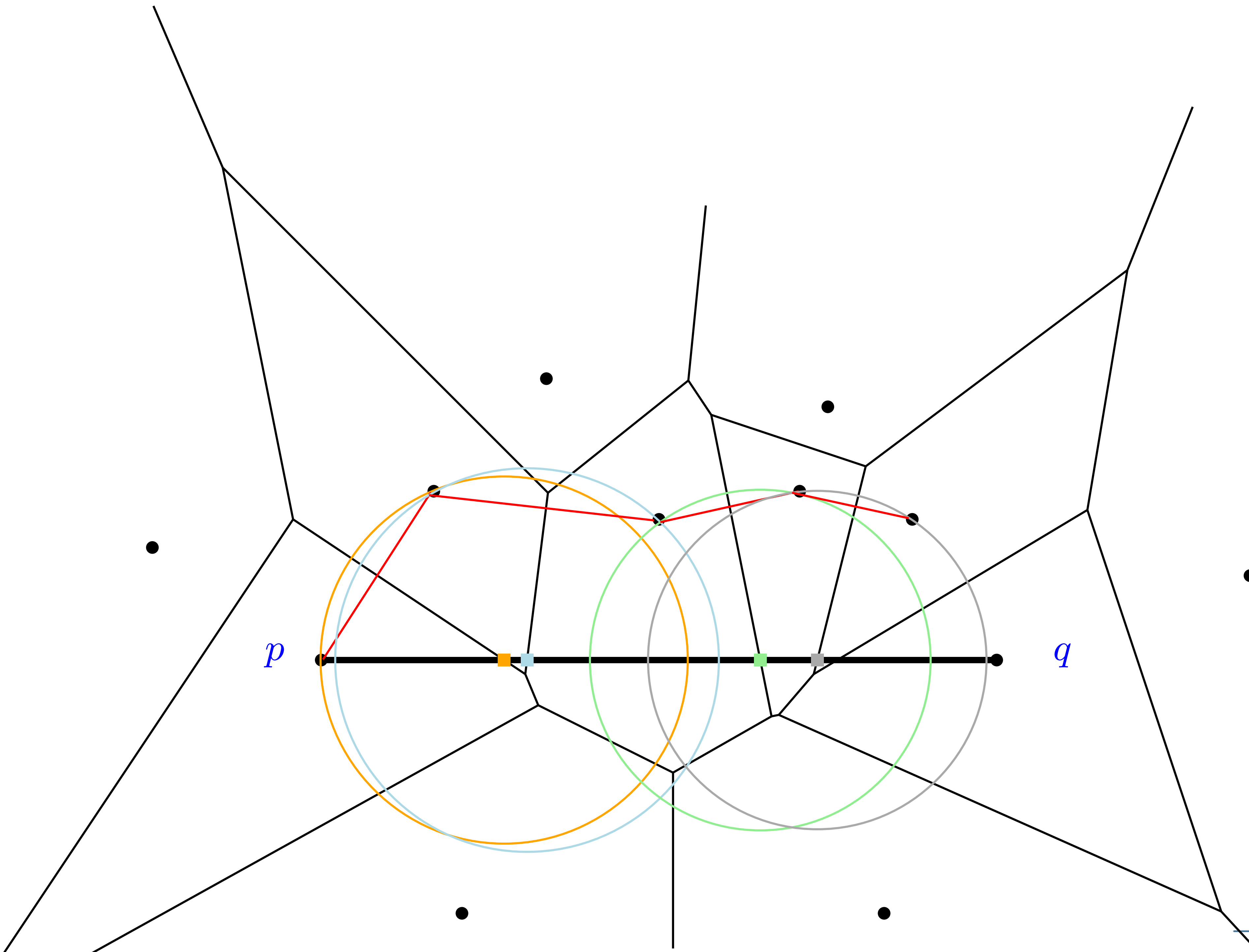

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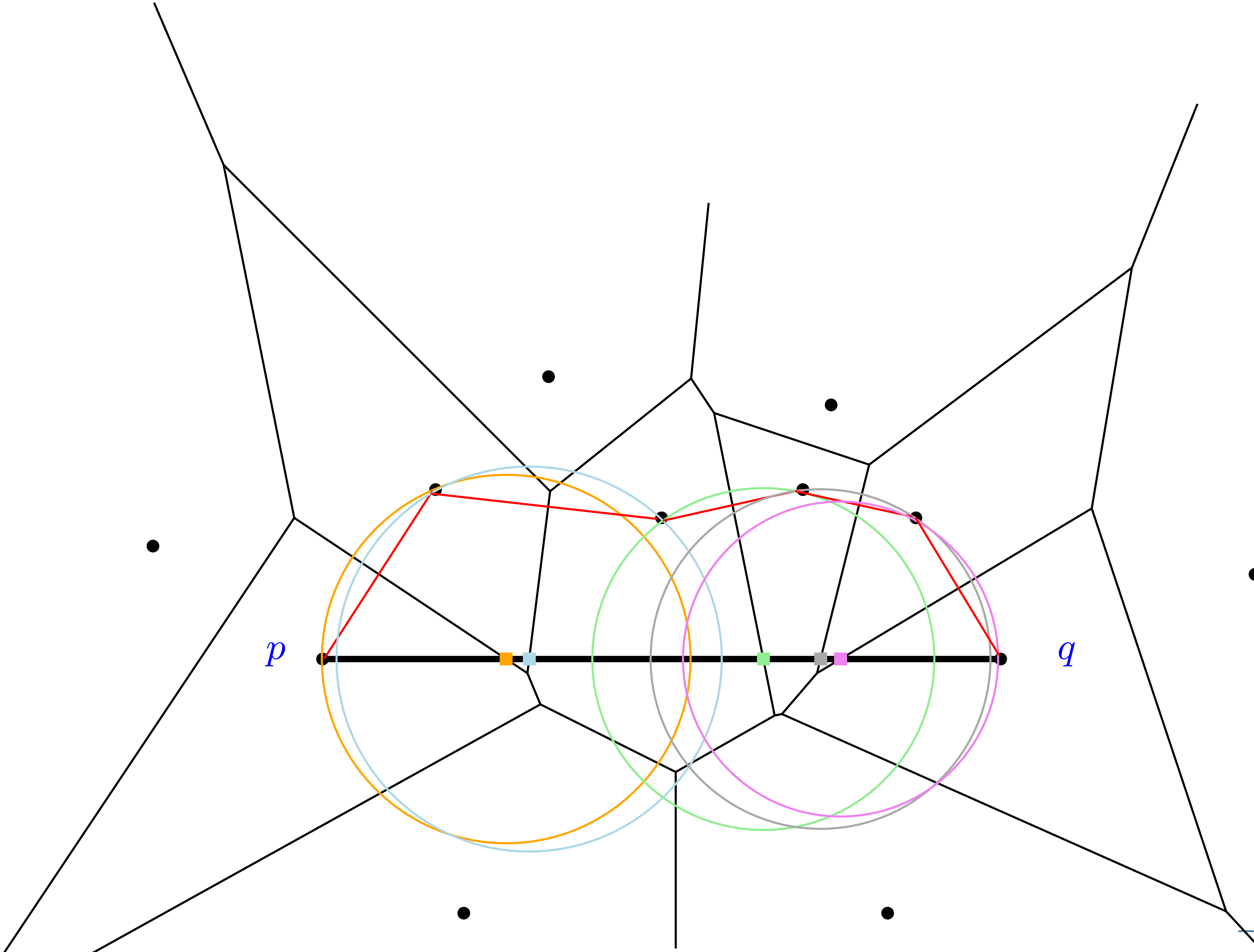


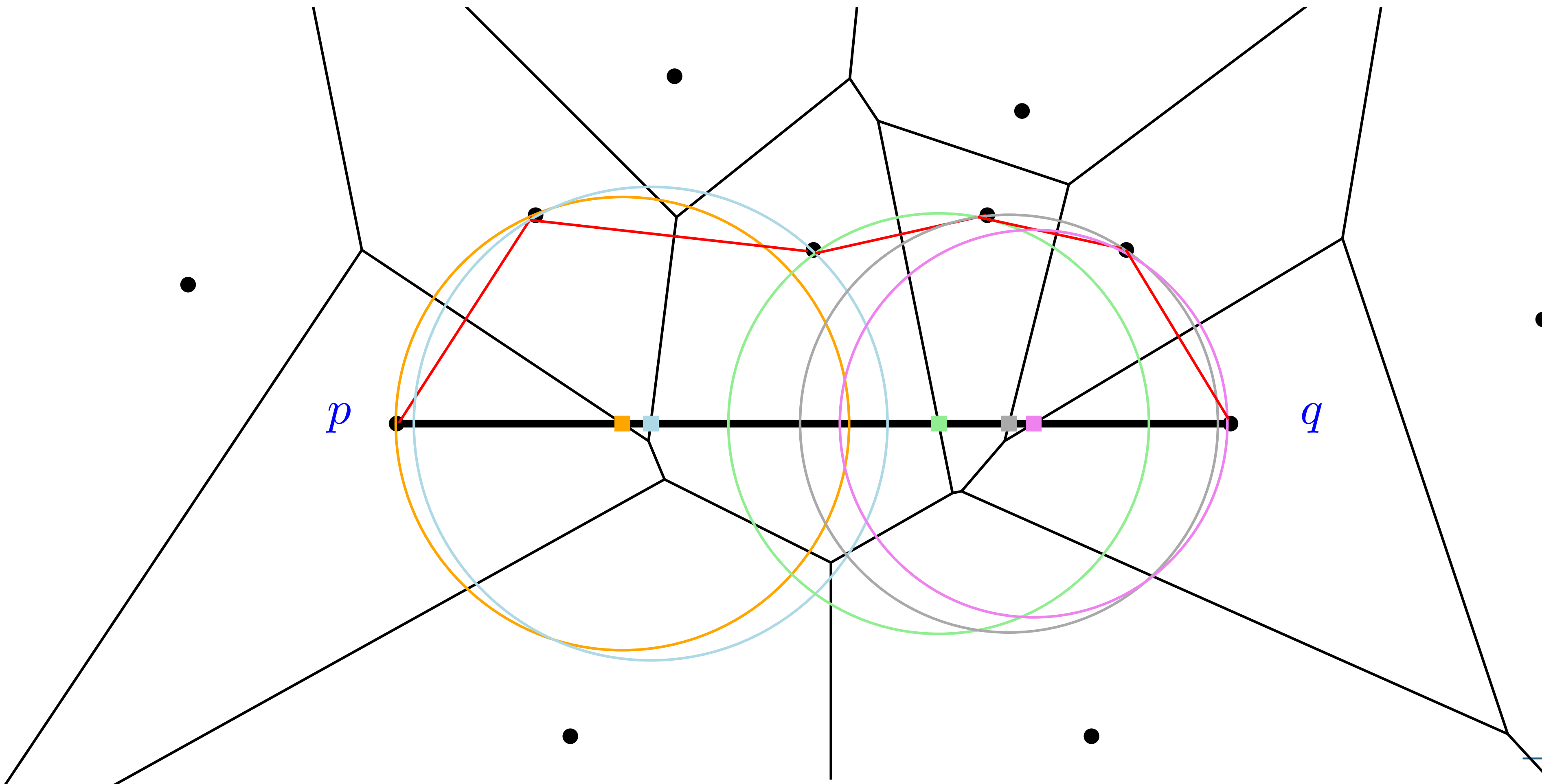
Delaunay triangulations as spanners

general case, where path crosses connecting segment is similar but more complicated

This method does not generalize to $d > 2$.

best proven stretch factor 1.998

Delaunay triangulations wirth respect to other metrics work as well or better



θ-graphs as geometric spanners

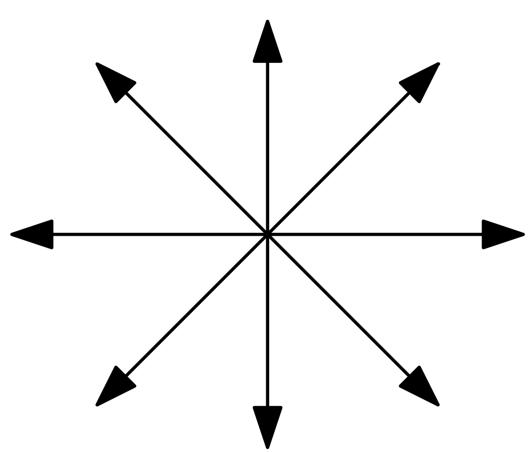
θ some angle, sufficiently small (e.g. less than $\pi/3$); $\phi = \theta/2$

Let U be a "small" set of directions, so that every possible direction has angle at most ϕ with some $u \in U$.

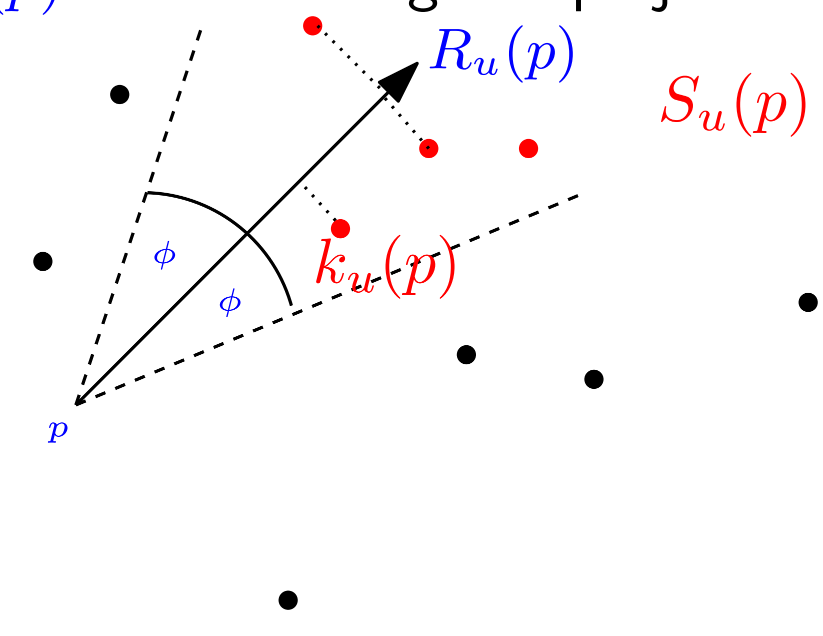
for point p and $u \in U$ let $R_u(p)$ be the ray in direction u starting at p

for point p and $u \in U$ let $S_u(p) = \{q \in S \setminus \{q\} \mid \angle(\vec{pq}, u) \leq \phi\}$

for point p and $u \in U$ let $k_u(p)$ be the point in $S_u(p)$ whose orthogonal projection onto ray $R_u(p)$ is closest to p.



$\phi = \pi/8$



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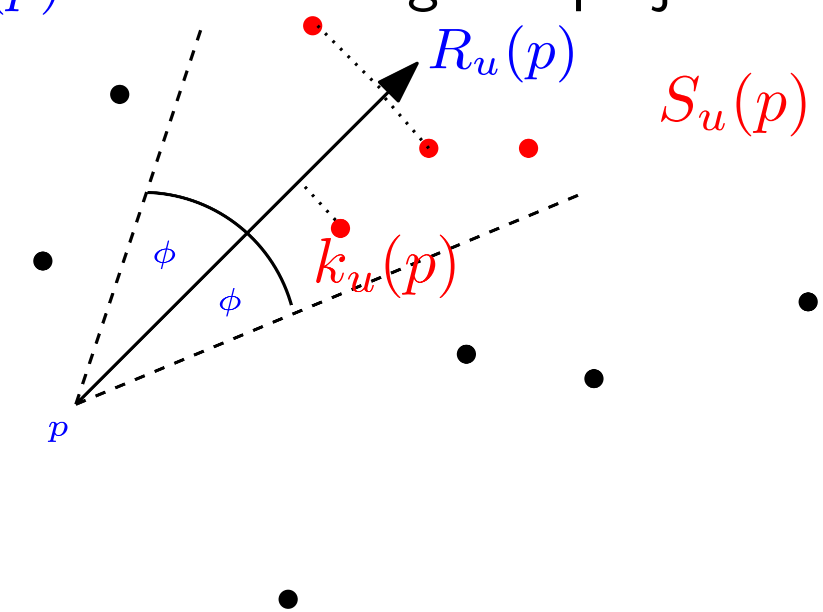
The θ-graph for S consists of the edges

$$\{\{p, k_u(p)\} \mid p \in S \text{ and } u \in U\}$$



θ-graphs as geometric spanners

Finding a short path from p to q in θ-graph:

$p_0 = p$; $i := 0$
while $p_i \neq q$ **do**
  let u_i be such that $q \in S_{u_i}(p_i)$
  $p_{i+1} = k_{u_i}(p_i)$
  $i := i + 1$

θ-graphs as geometric spanners

Lemma: Let $\delta_i = \delta(p_i, p_{i+1})$ and let $\ell_i = \delta(p_i, q)$.

$$\delta_i + \ell_{i+1} \leq \ell_i + 2\delta_i \sin\phi$$

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Corollary:

$$\sum_{0 \leq i < m} \delta_i \leq \frac{\delta(p,q)}{1 - 2\sin\phi}$$

The θ-graph is a t-spanner with $t \leq \frac{1}{1-2\sin(\theta/2)}$ and $\lceil 2\pi/\theta \rceil \cdot |S|$ edges.

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Geometric spanners from WSPD

Well **S**eparated **P**air **D**ecomposition for a set S of n points

Geometric spanners from WSPD

Well **S**eparated **P**air **D**ecomposition for a set S of n points with parameter $1/\varepsilon$

sequence of pairs of subsets of S: (A_i, B_i) with $i = 1, \ldots, s$ with

1. $A_i \cap B_i = \emptyset$ for each i
2. for every pair $p, q \in S$ there is exactly on pair (A_i, B_i) s.t. $p \in A_i$ and $q \in B_i$ (or vice versa)
3. for each i the sets A_i and B_i are $(1/\varepsilon)$-separated.

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the largest distance between points in the same set is at most ε time the smallest distance between points from different sets.

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Theorem: Given a set S of n points in $\mathbb{R}^d$ and a parameter $\varepsilon > 0$ via a WSPD for S you can compute a $(1+\varepsilon)$-spanner for S with $O(n/\varepsilon^d)$ edges in time $O(n \log n + n/\varepsilon^d)$.

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Sketch of proof:
let $c \geq 16$ and $\delta = \varepsilon/c$.

Compute a $(1/\delta)$-WSPD for S and for every pair (u, v) in the decomposition take edge $\{\mathrm{rep}_u, \mathrm{rep}_v\}$

Computing a WSPD

1. Compute a quadtree (octtree) T for S (compressed)
2. Execute CompWSPD(root(T),root(T),T), where

CompWSPD(u, v, T)
  if $\Delta(u) < \Delta(v)$ **then** exchange u and v
  if $\Delta(u) \leq \varepsilon \cdot \delta(u, v)$ **then return** $\{\{u, v\}\}$

  return $\bigcup_{w \text{ childof } u}$ CompWSPD(w, v, T)