Packing and covering: planar separator and shifting

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Computaional Geometry
Summer semester 2020

max planck institut
informatik

Overview

- Planar separator theorem (slides by Mark de Berg)

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- Indepedent set in planar graphs (slides by MdB)

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- Exact algroithms for packing and covering

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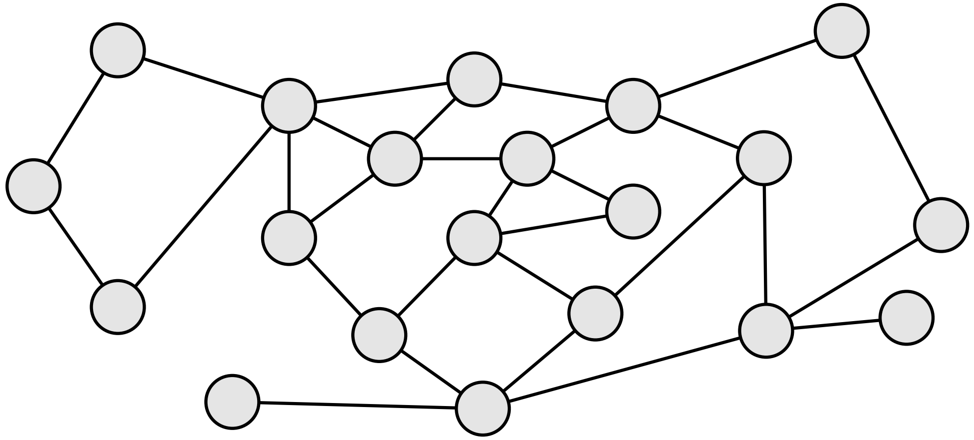
- Indepedent set in planar graphs (slides by MdB)
- Exact algroithms for packing and covering
- Shifting strategy: approximation schemes

Planar graphs

Planar graphs: graphs that can be drawn without crossing edges



slide by Mark de Berg

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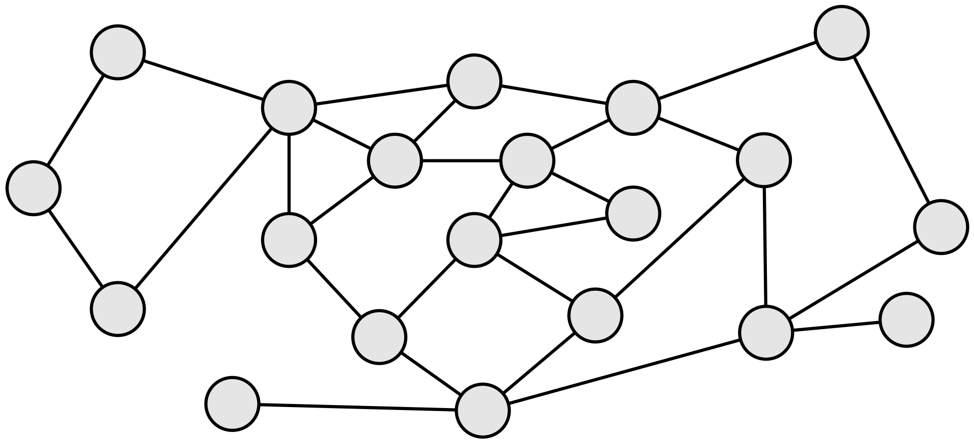

Planar Separator Theorem (Lipton,Tarjan 1979)

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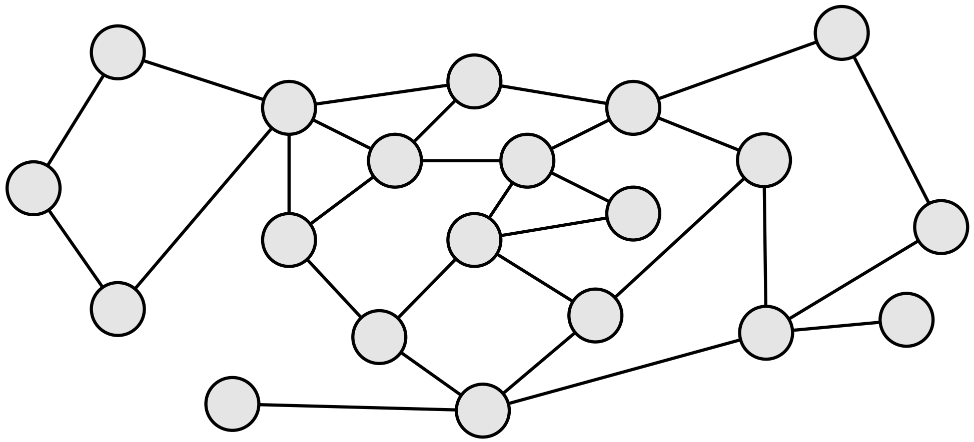

Planar Separator Theorem (Lipton,Tarjan 1979)

For any planar graph $G = (V, E)$ there is a separator $S \subset V$ of size $O(\sqrt{n})$ such that $V \setminus S$ can be partitioned into subsets A and B, each of size at most $\frac{2}{3}n$ and with no edges between them.

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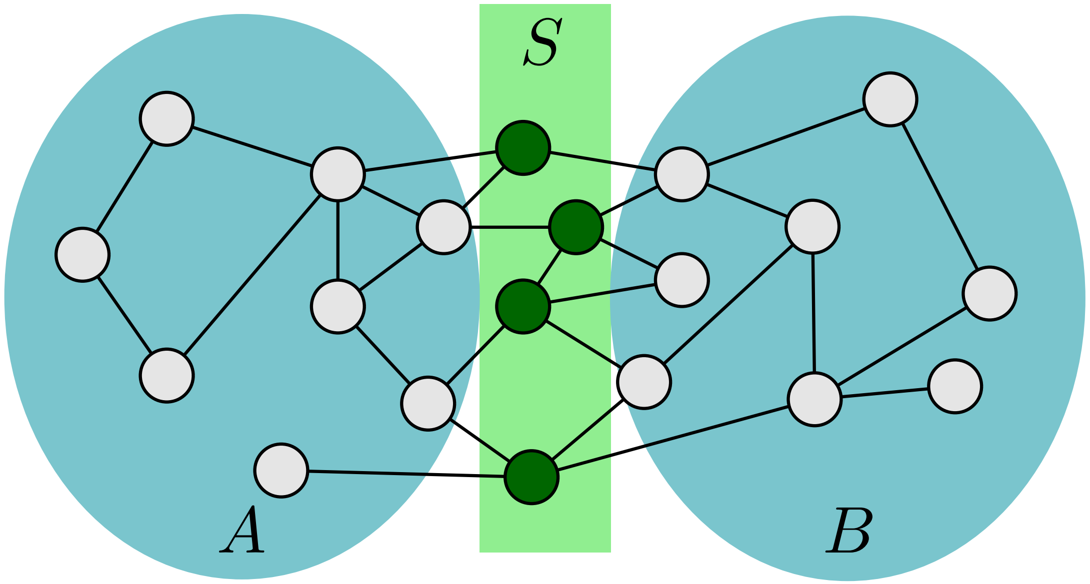

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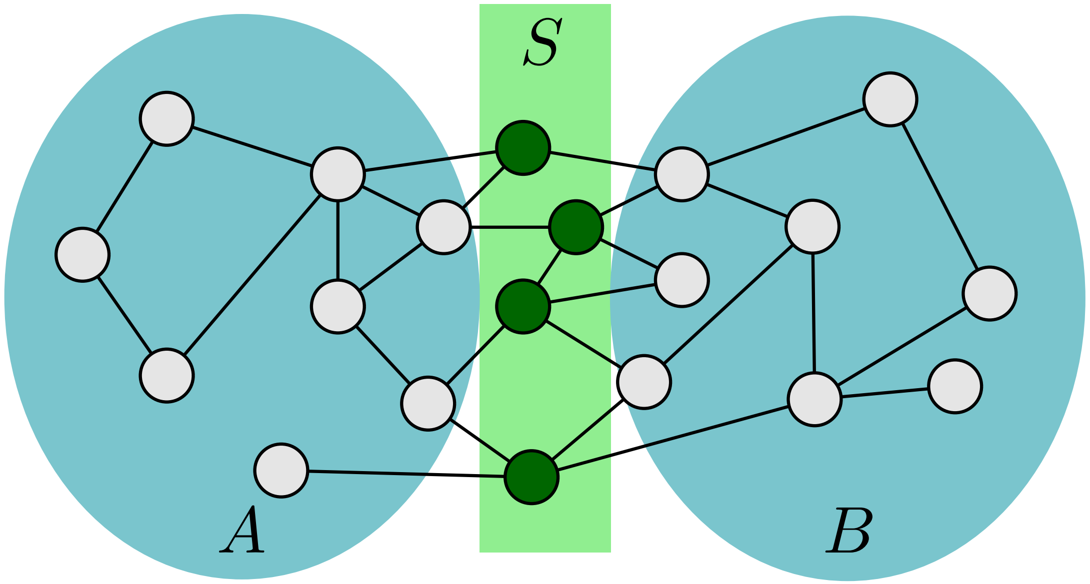

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Such a $(2/3)$-balanced separator can be computed in $O(n)$ time.

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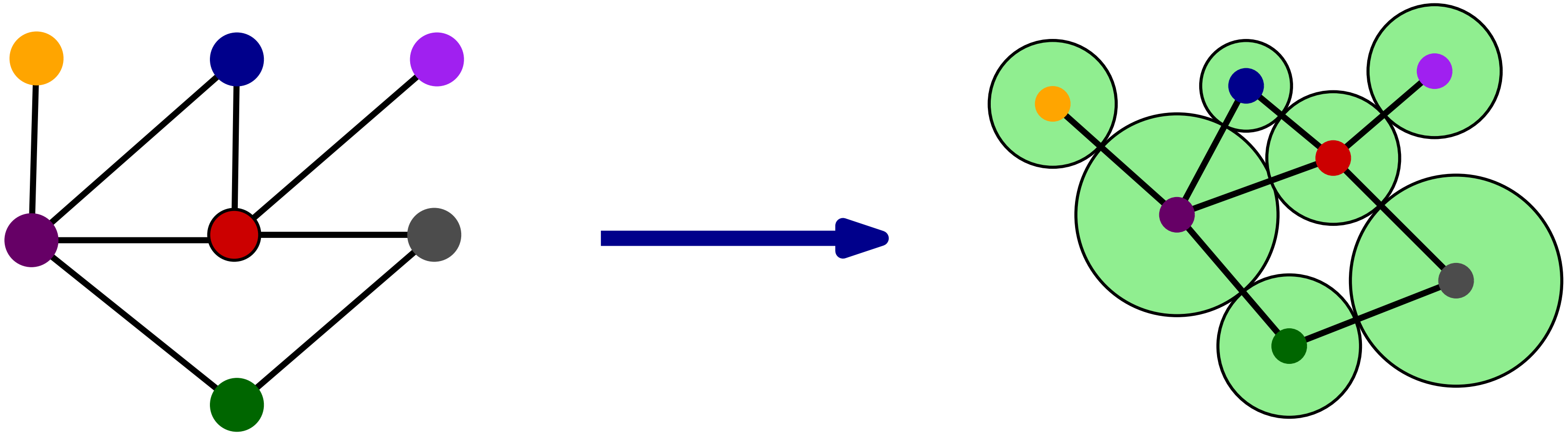
A geometric proof of the Planar Separator Theorem

Fact: Any planar graph is the contact graph of a set of interior-disjoint disks.



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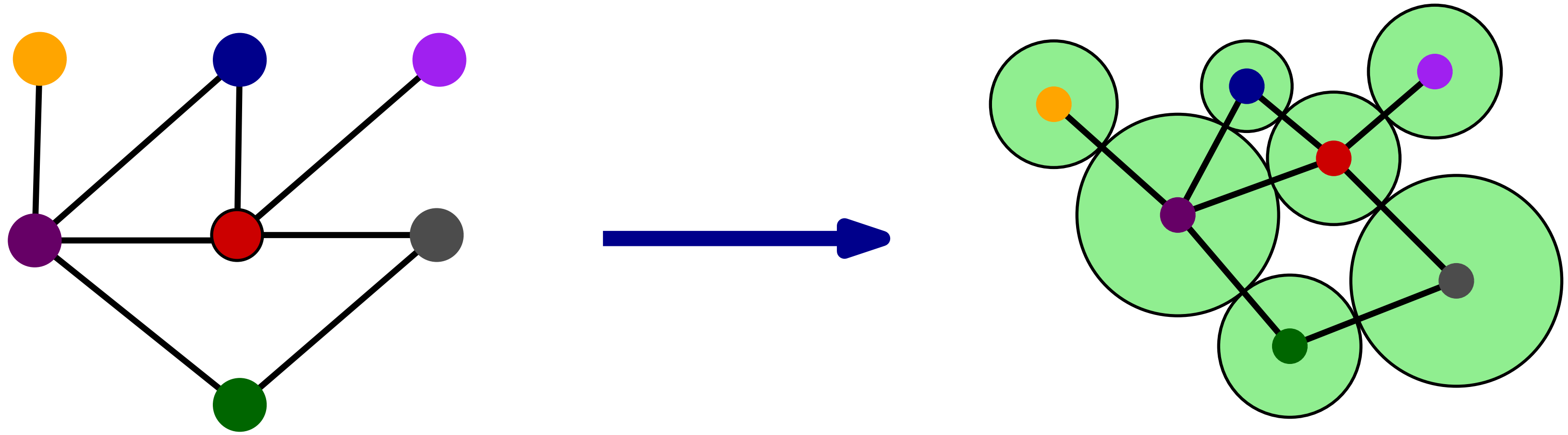
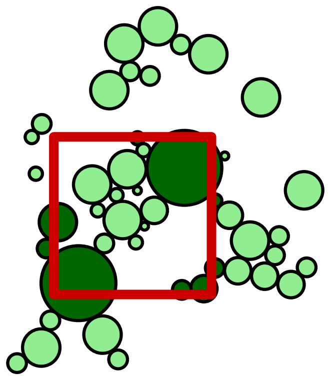


Proof idea: Find a square σ intersecting $O(\sqrt{n})$ disks that is a balanced separator.



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Theorem. For any contact graph of n interior-disjoint disks, there is an α-balanced separator of size $O(\sqrt{n})$, where $\alpha = 36/37$.

Proof.

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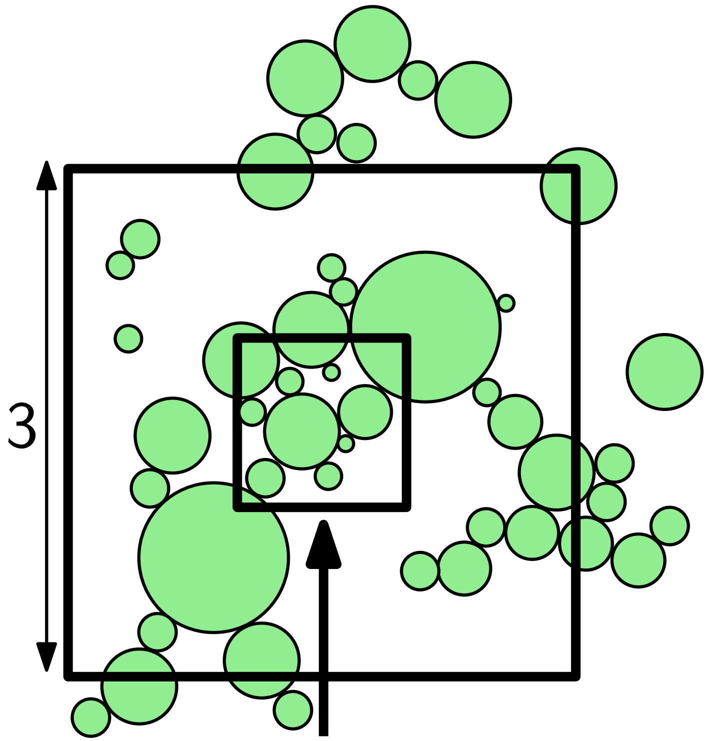

smallest square containing at least $n/37$ disks

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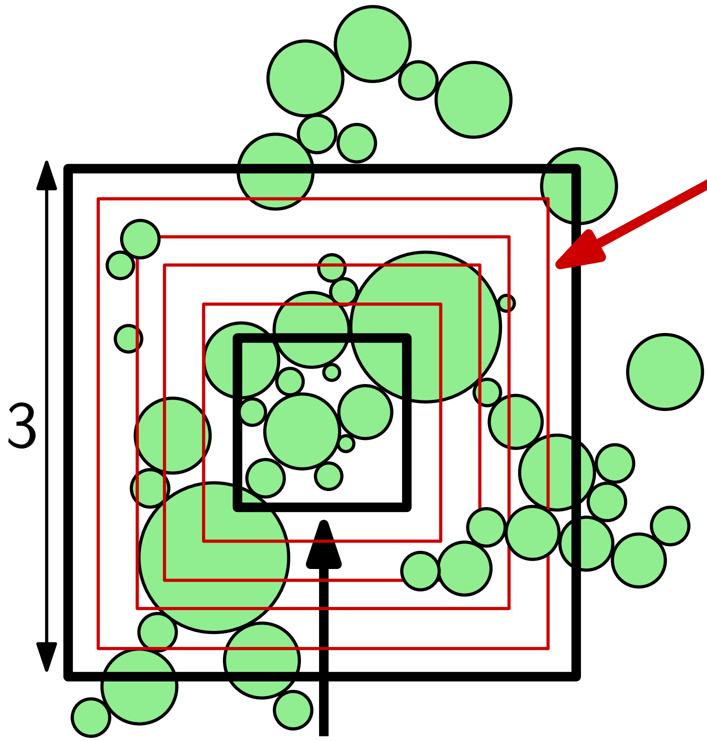

$\sqrt{n}$ squares $\sigma_1, \ldots, \sigma_{\sqrt{n}}$
at distance $1/\sqrt{n}$ from each other

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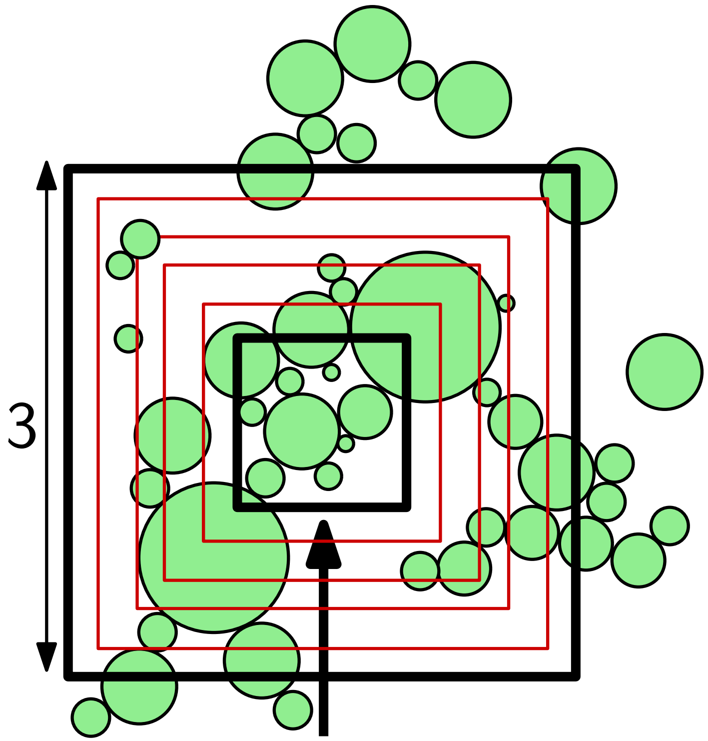

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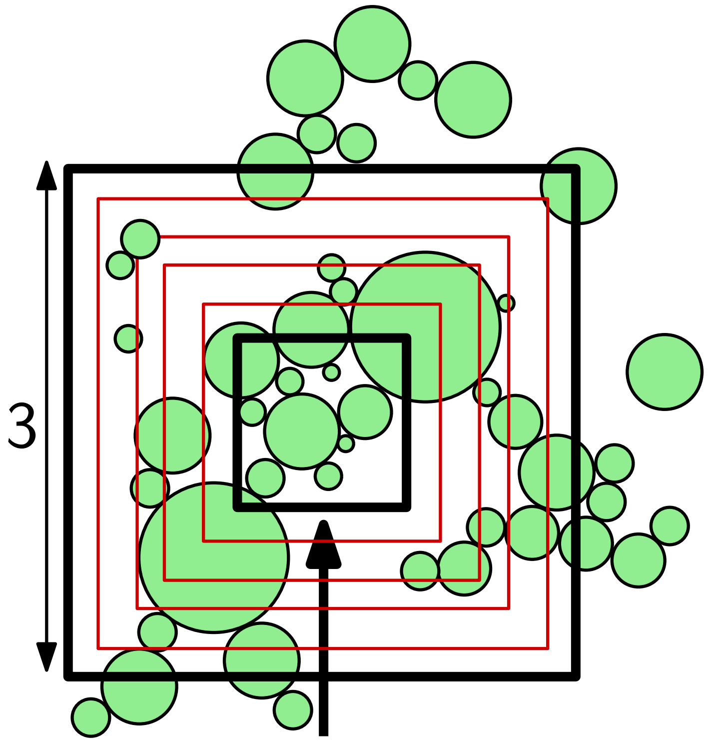

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Constructing the separator:
Select a square σ_i that intersects $O(\sqrt{n})$ disks and put these disks into the separator.

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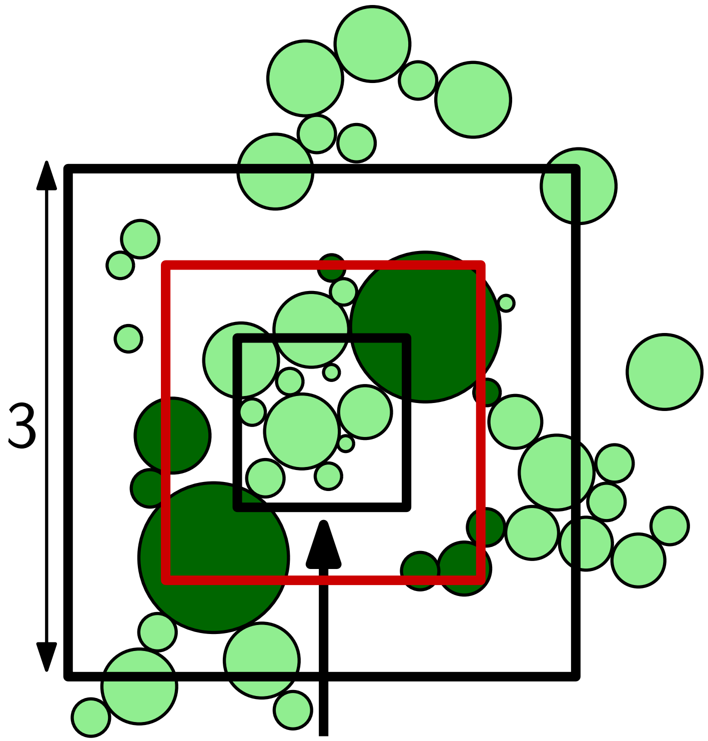

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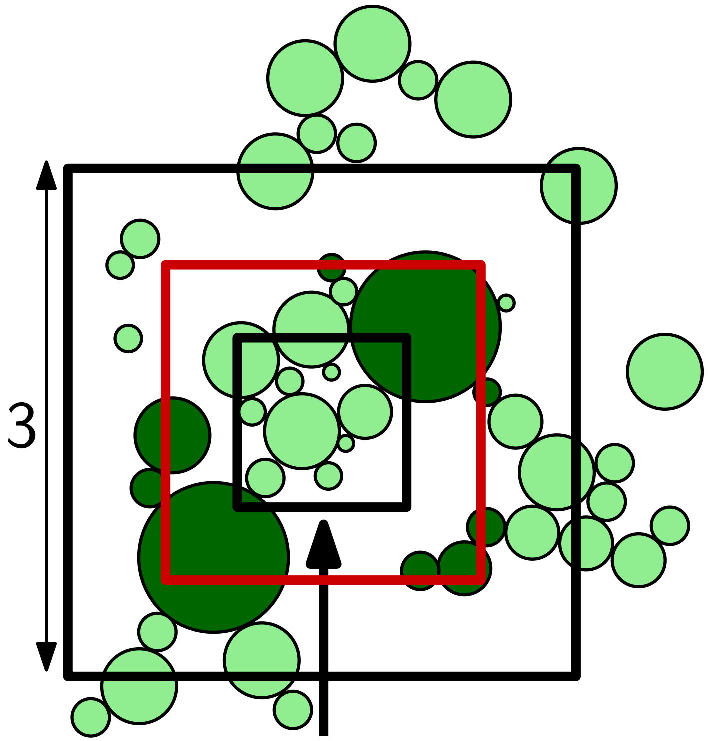

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Things to check

- separator is $(36/37)$-balanced
- does square σ_i with the desired property actually exist ??

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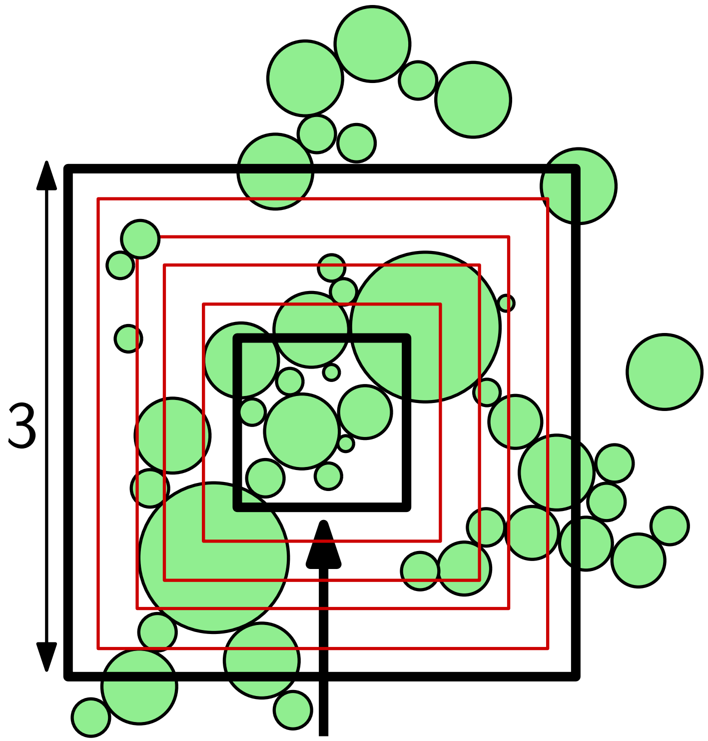

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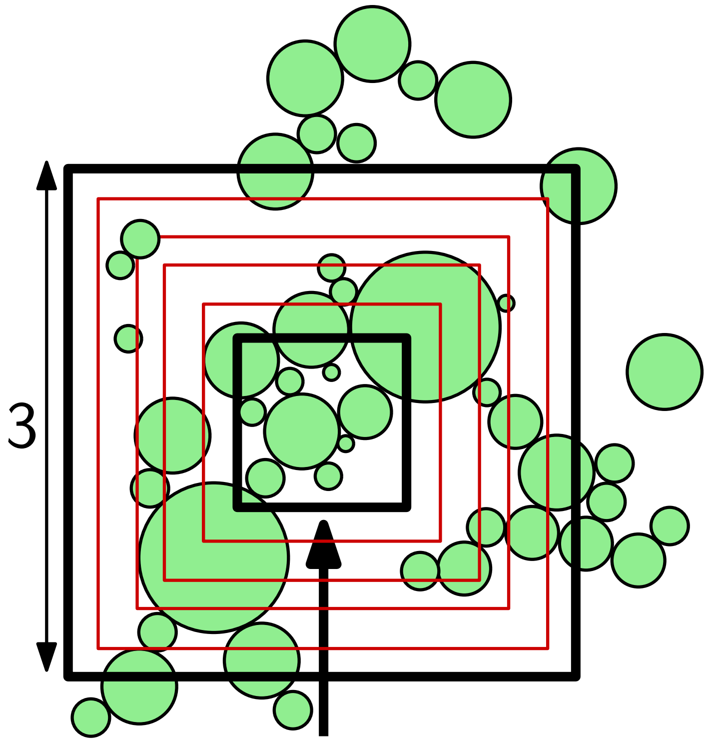
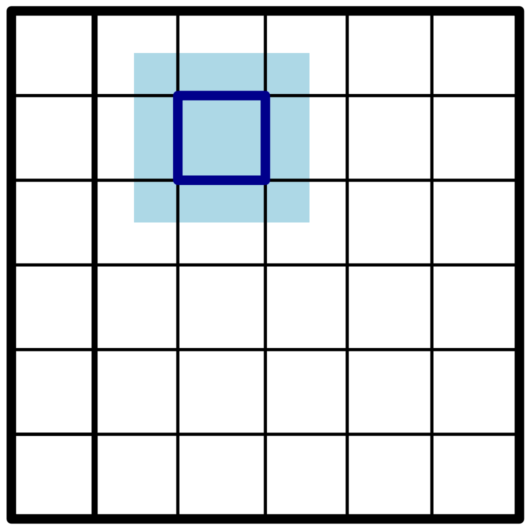


smallest square containing at least $n/37$ disks

separator is $(36/37)$-balanced

- at least $n/37$ disk inside
- at most $36n/37$ disks inside



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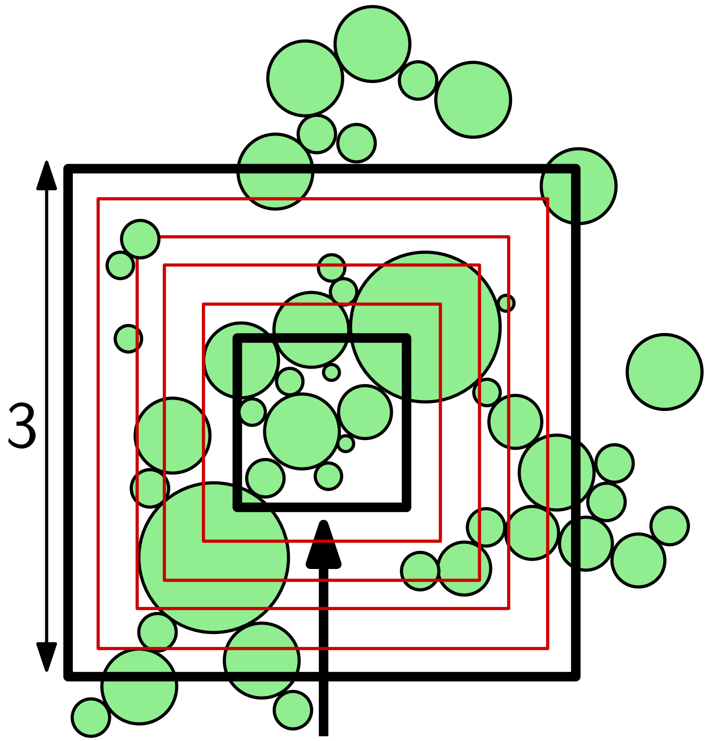

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Does σ_i intersecting $O(\sqrt{n})$ disks exist?

total number of disk-square intersections

$$\leqslant \sum_{i=1}^{n_{\text{small}}} (1 + \text{diam}(D_i) \cdot \sqrt{n})$$

$$\leqslant n_{\text{small}} + O(\sqrt{n}) \cdot \sum_{i=1}^{n_{\text{small}}} \sqrt{\text{area}(D_i)}$$

$$= O(n)$$

last step uses

- $\sum_{i=1}^{n_{\text{small}}} \text{area}(D_i) = O(1)$ (sort of ...)
- $\sum_{i=1}^{k} \sqrt{a_i} \leqslant \sum_{i=1}^{k} \sqrt{\frac{\sum_{i=1}^{k} a_i}{k}}$

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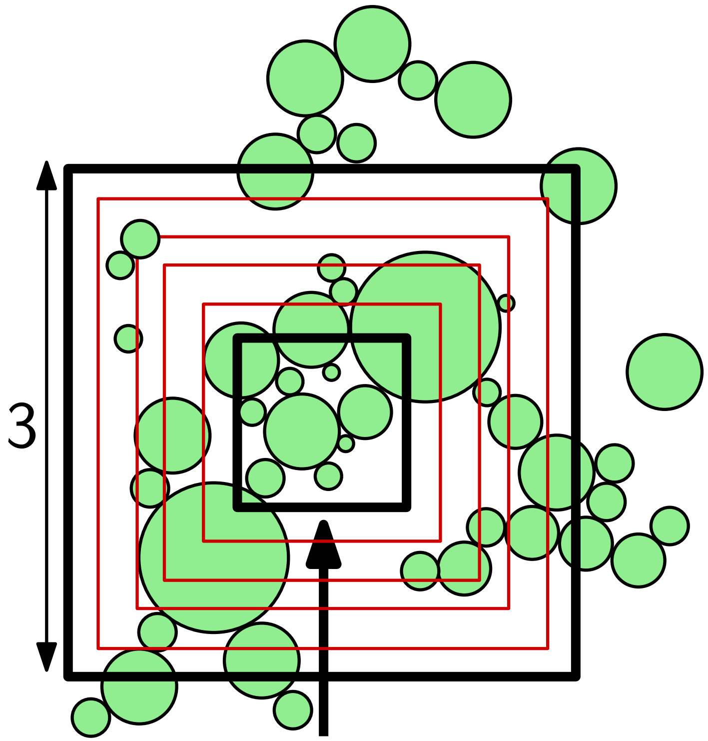

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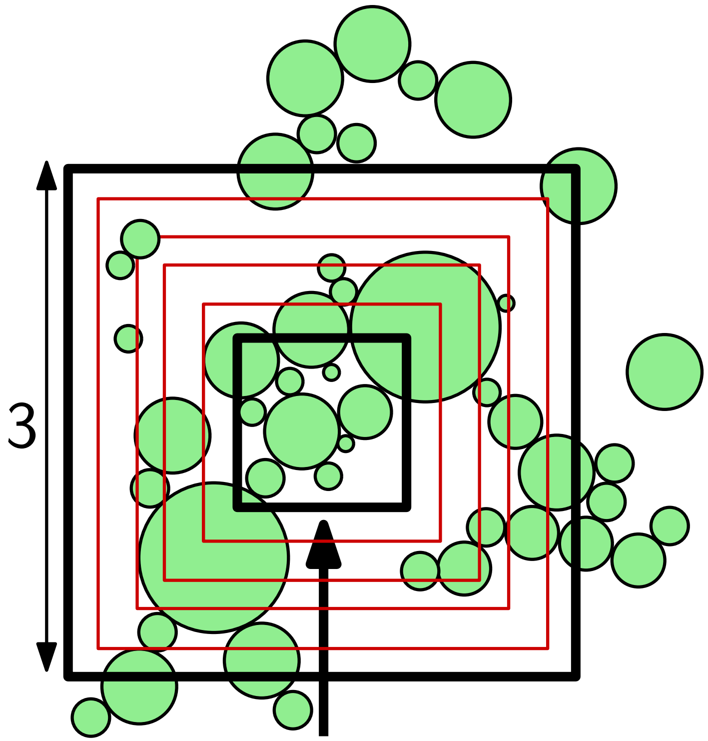

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$\Longrightarrow$ one of the σ_i's intersects $O(\sqrt{n})$ disks

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Subexponential algorithms on planar graphs

Theorem. INDEPENDENT SET can be solved in $2^{O(\sqrt{n})}$ time in planar graphs.

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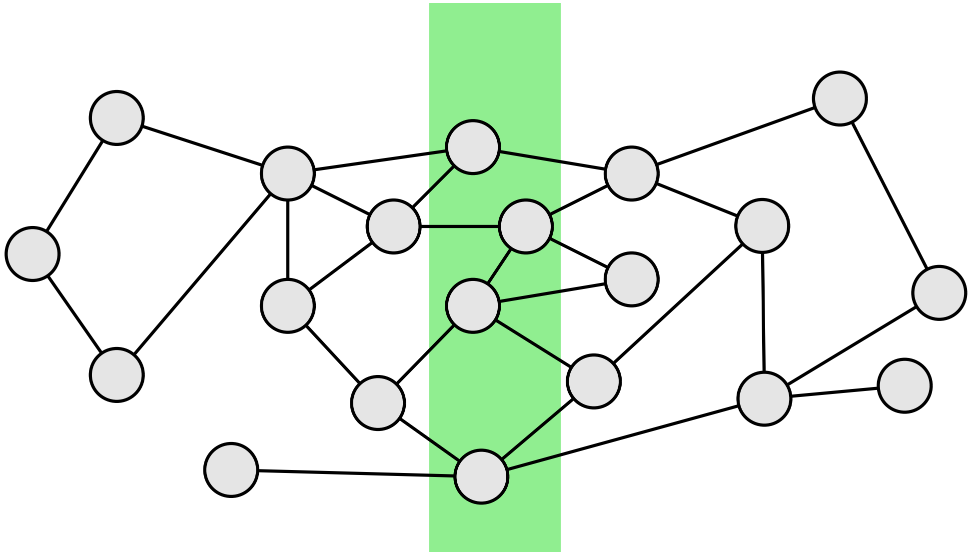

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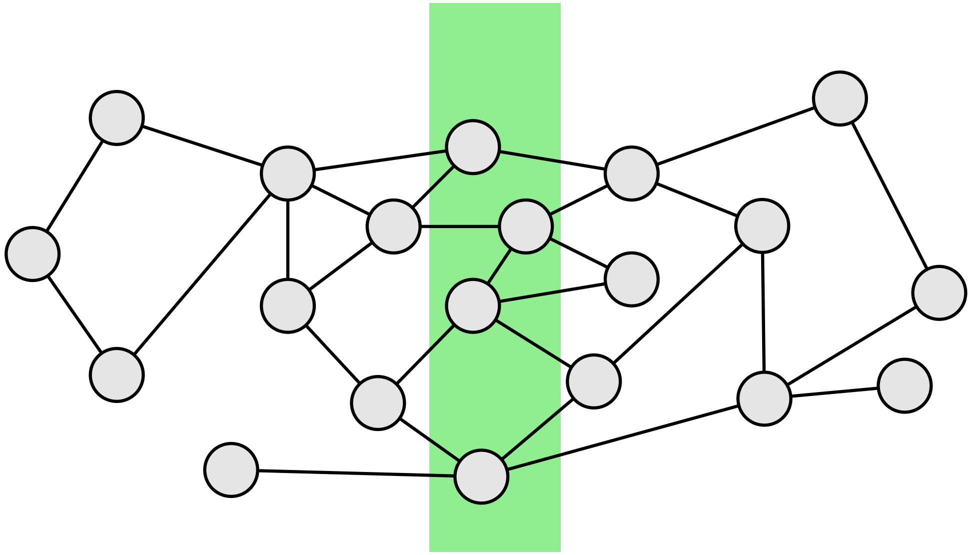

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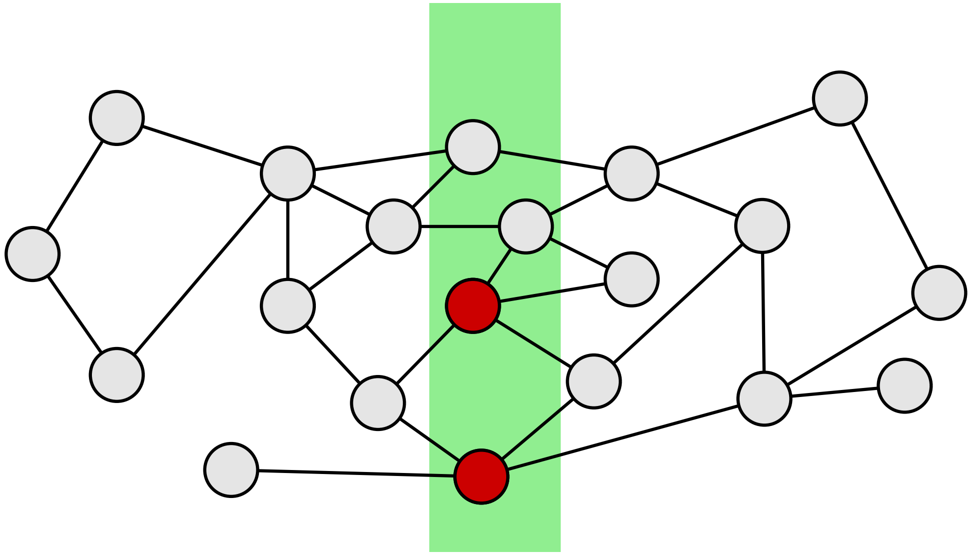

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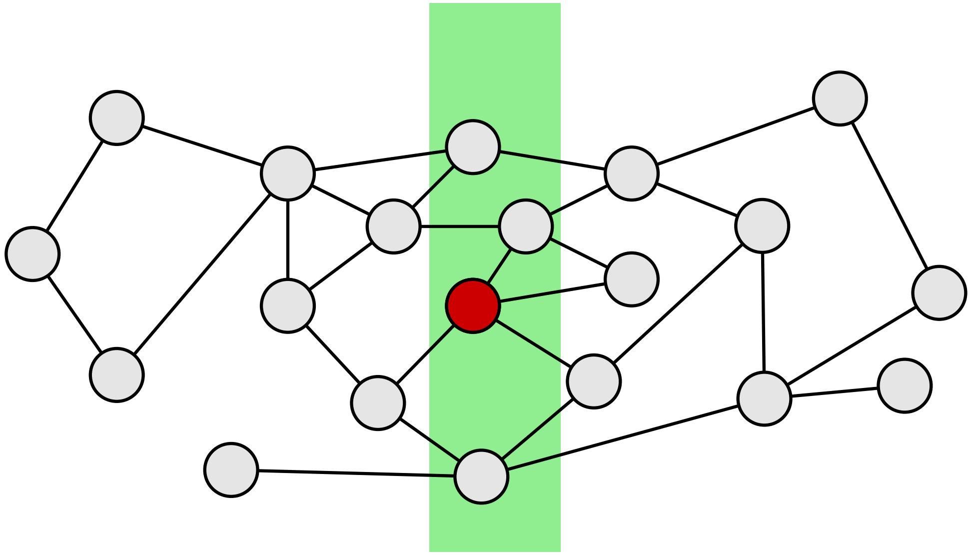

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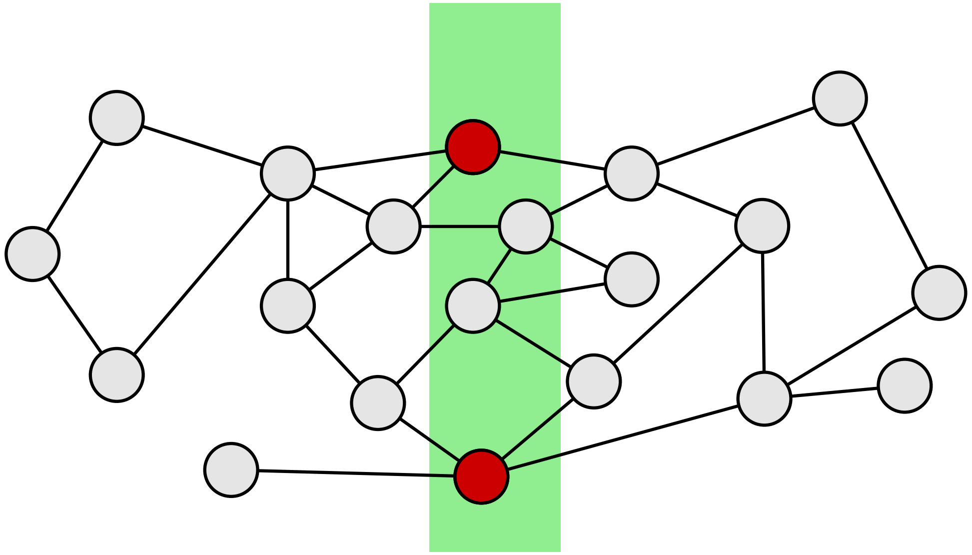

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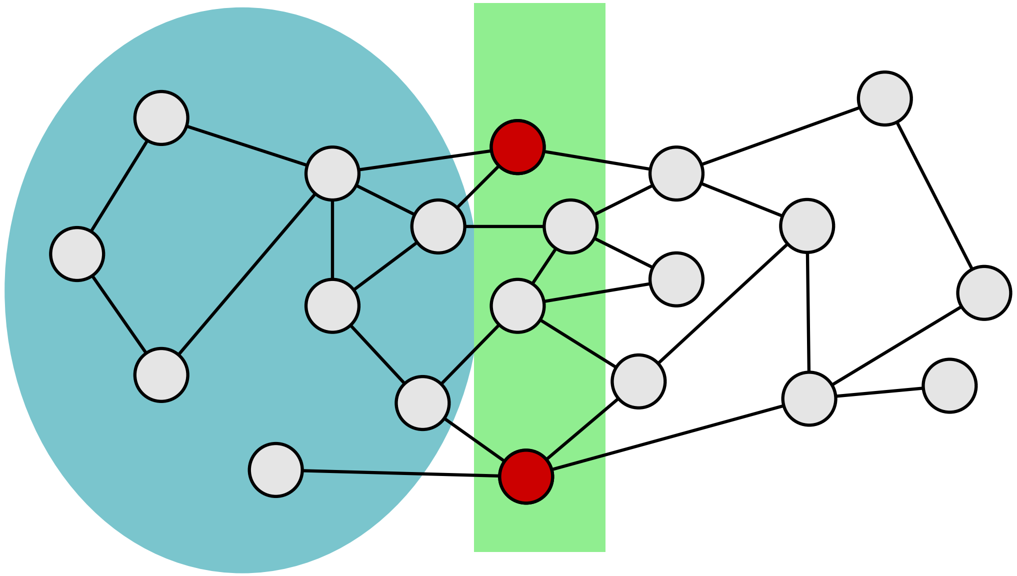

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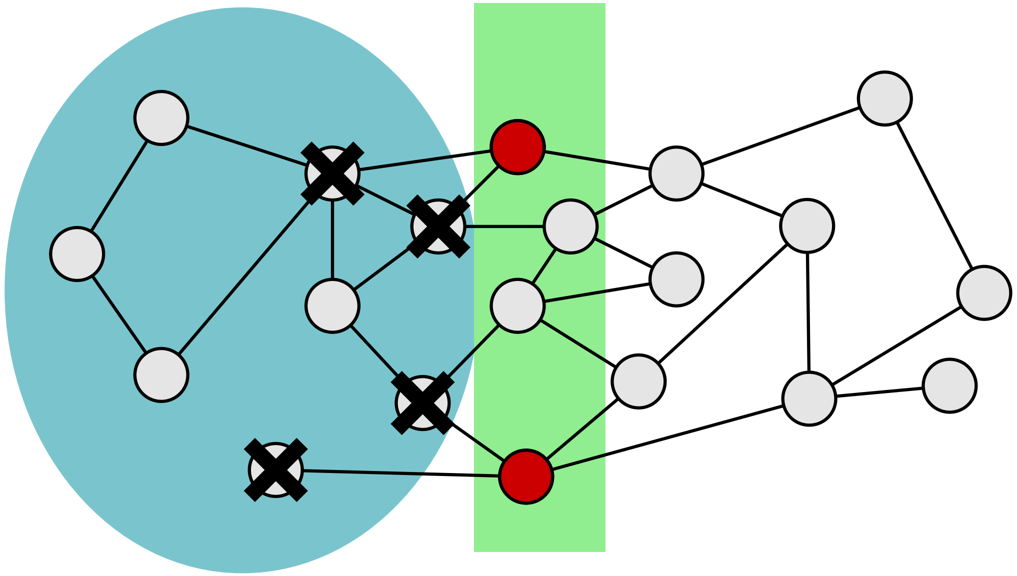

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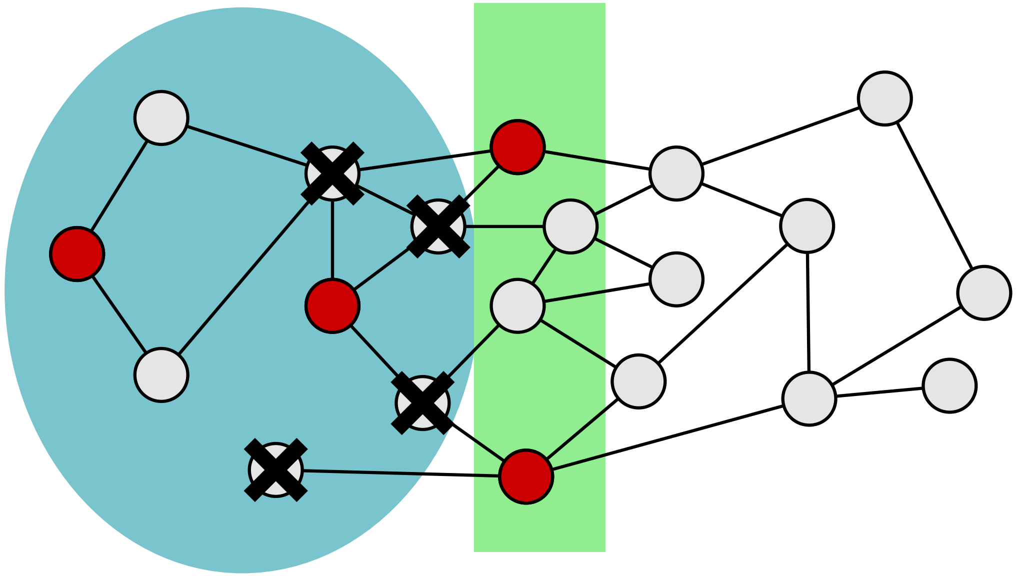

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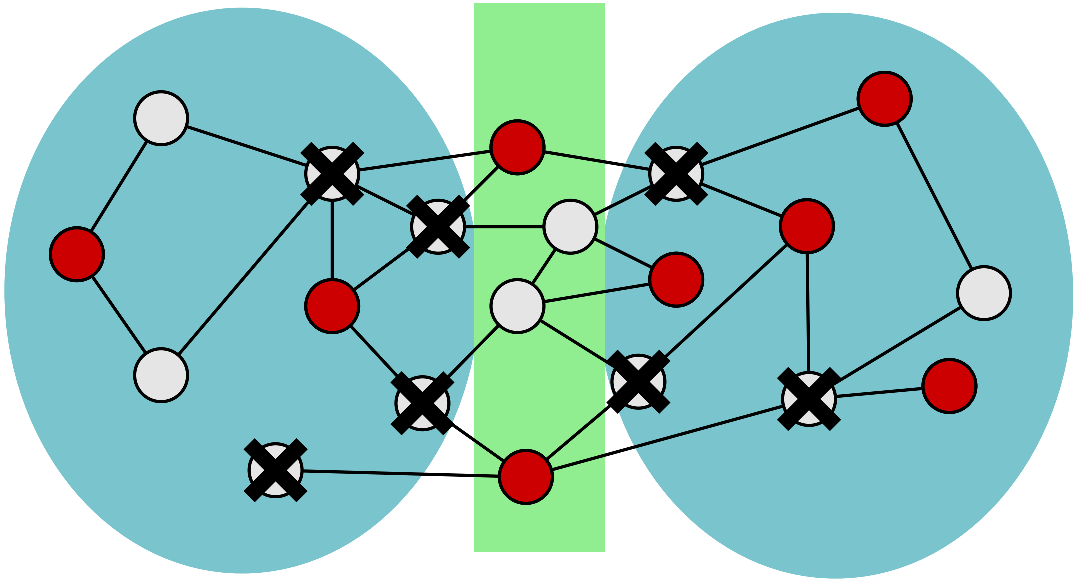

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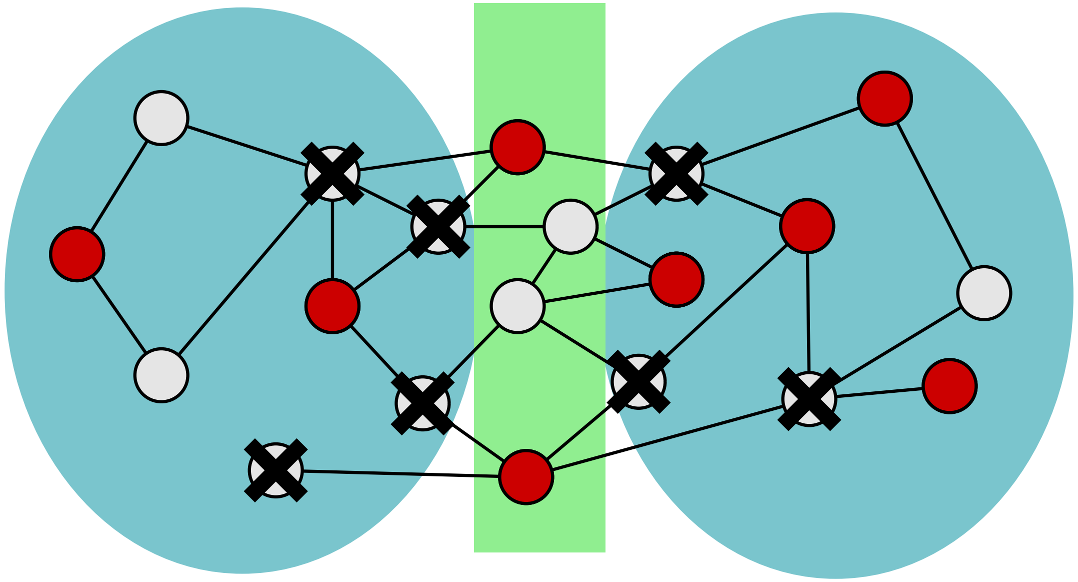

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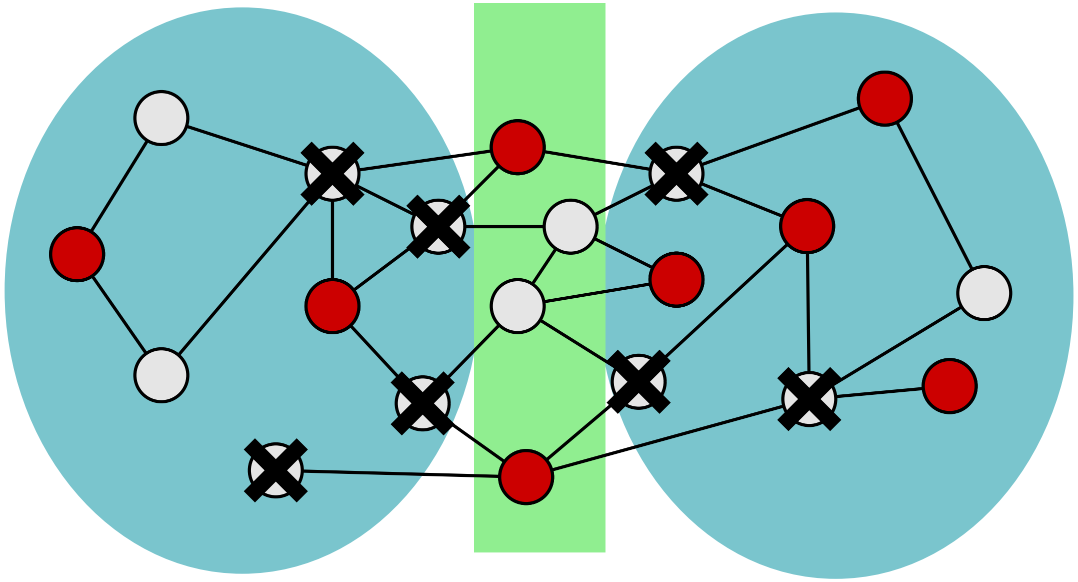

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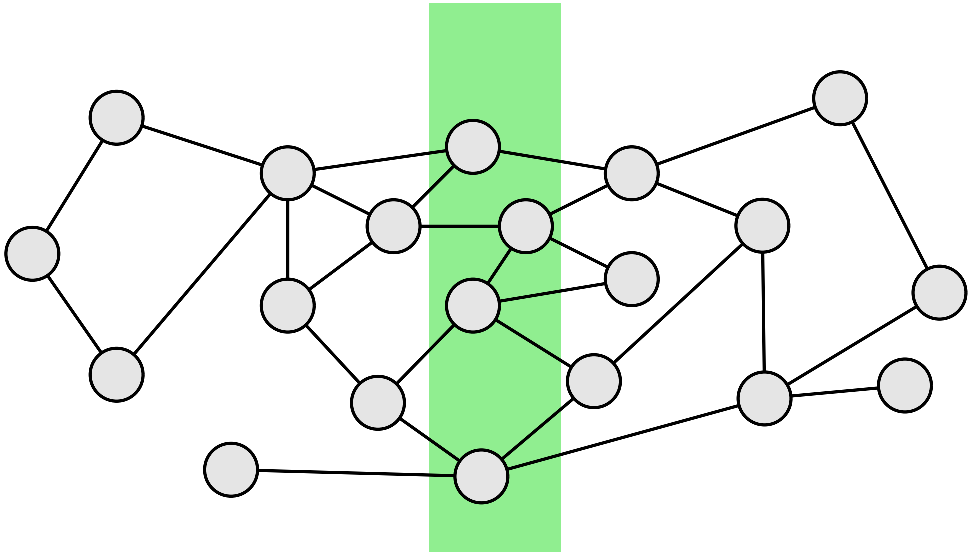

Running time

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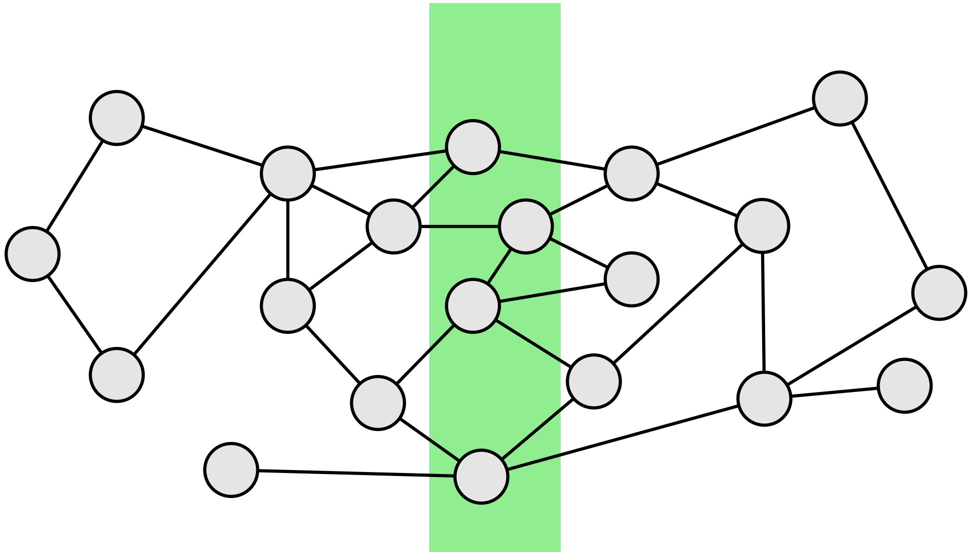

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$$T(n) \leqslant O(n) + 2^{O(\sqrt{n})} \cdot T(2n/3) \quad \Longrightarrow \quad T(n) = 2^{O(\sqrt{n})}$$

slide by Mark de Berg

Overview

- Planar separator theorem (slides by Mark de Berg)

- Indepedent set in planar graphs (slides by MdB)

- Exact algroithms for packing and covering

- Shifting strategy: approximation schemes

Intersection graphs

Given a set S of n objects in $\mathbb{R}^d$, their *intersection graph* has vertex set S and edge set

$$E[S] := \{ss' \mid s, s' \in S \text{ and } s \cap s' \neq \emptyset\}$$

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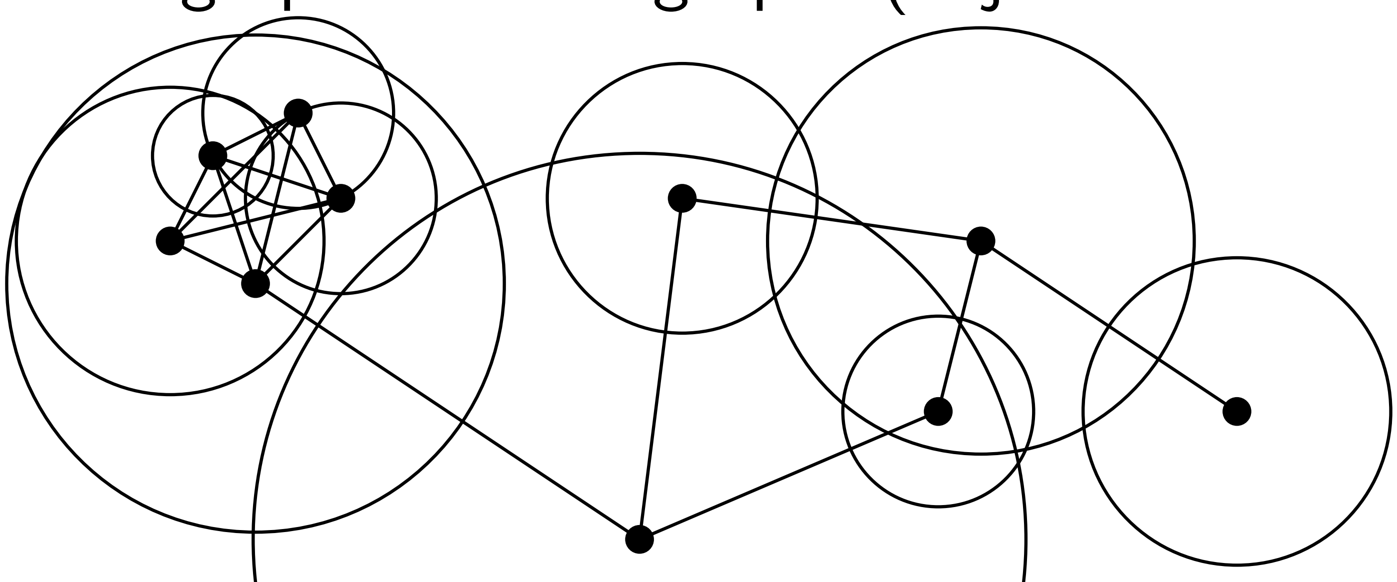
Planar graphs $\subset$ Disk graphs (object: disks in $\mathbb{R}^2$)



Packing: discrete vs continuous

Continuous:

Given n objects, do they fit in some other object without overlap?

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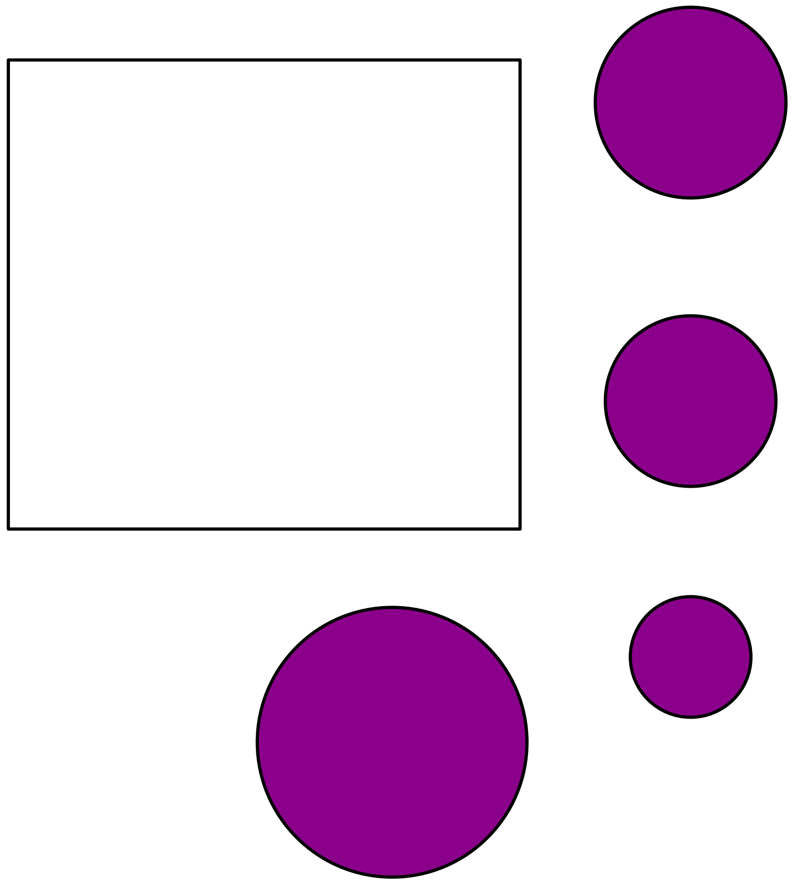

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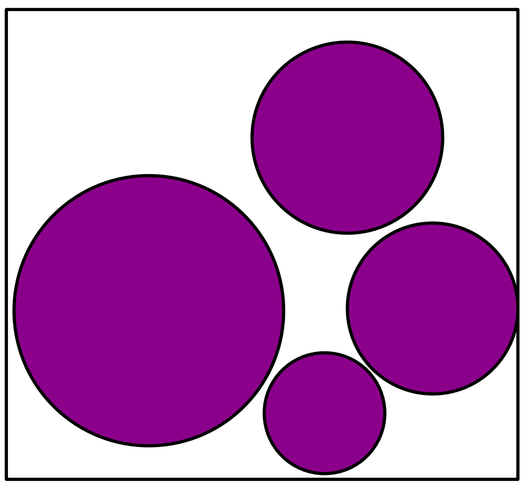

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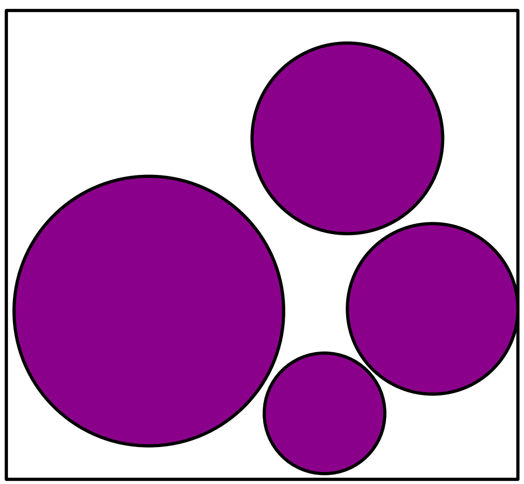

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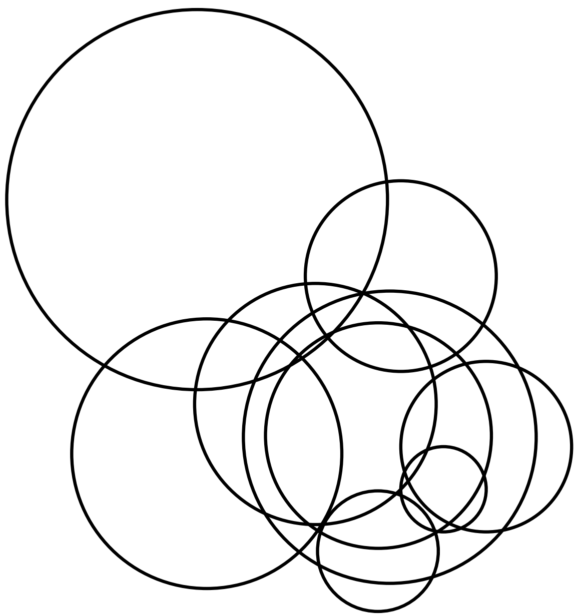
Given n objects, find maximum subset of non-overlapping objects



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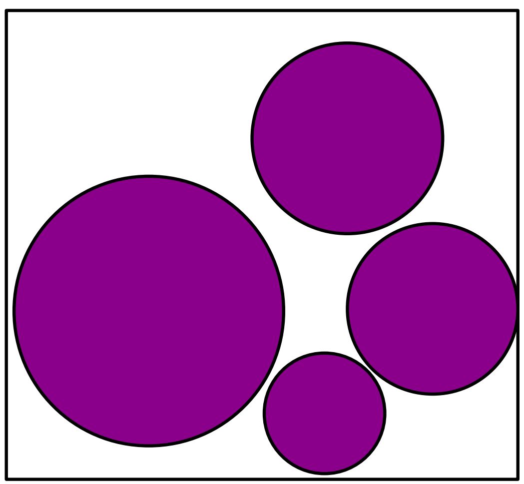

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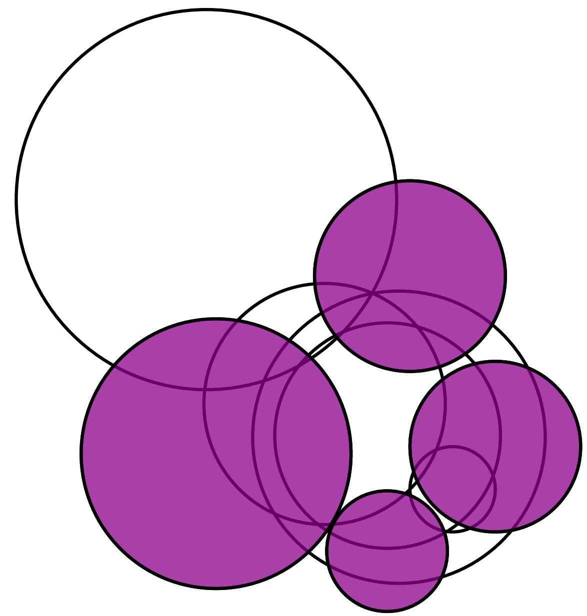

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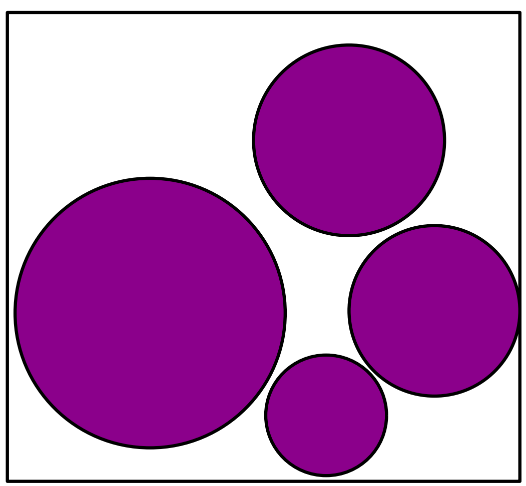

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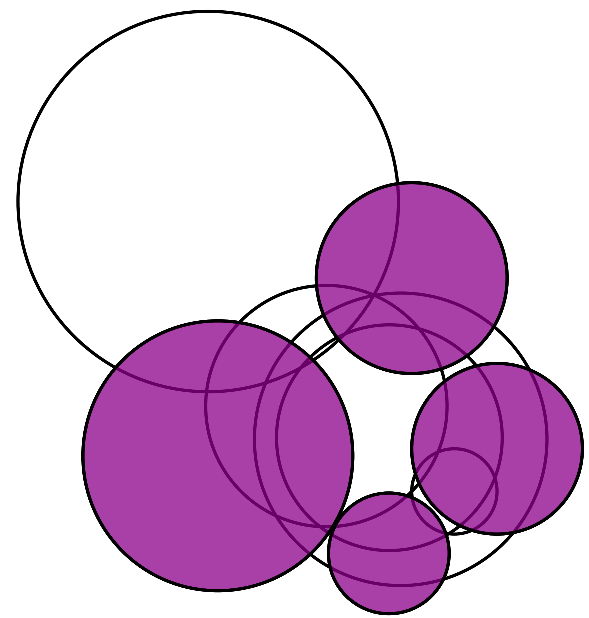

Same as max. independent set in intersection graph

Exact algorithm for discrete packing

Theorem. Independent set in intersection graphs of disks can be solved in $n^{O(\sqrt{k})}$ time, where $k =$ size of max indep. set.

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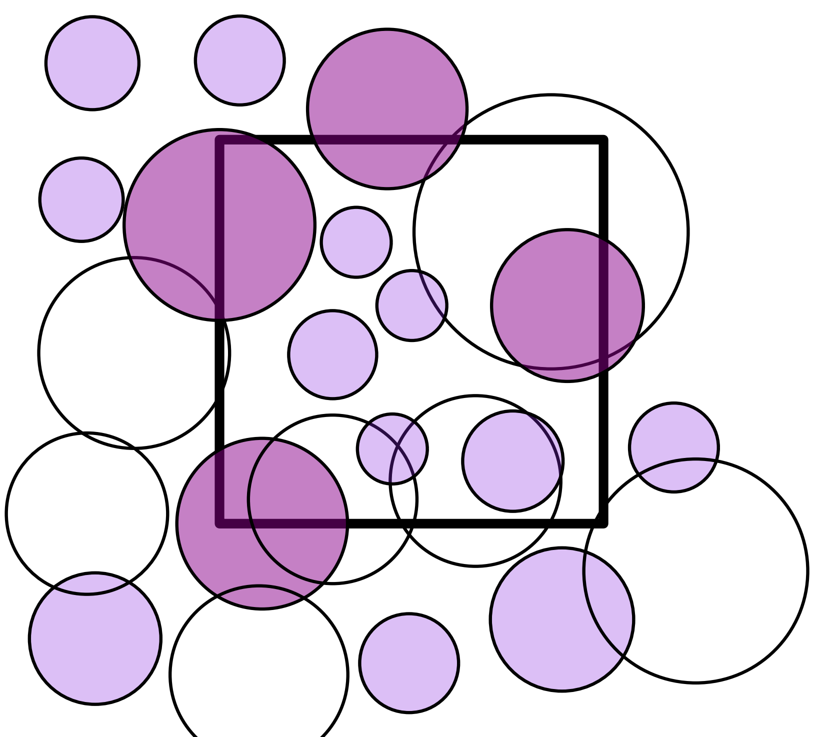
Solution I has k interior-disjoint disks.

There is a balanced separator square σ intersecting $O(\sqrt{k})$ disks from I.



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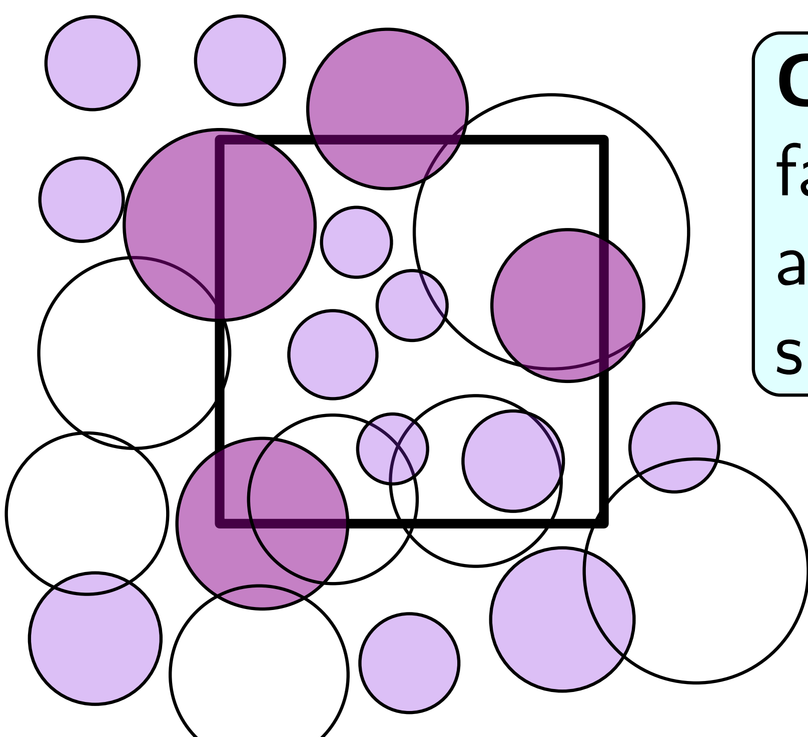
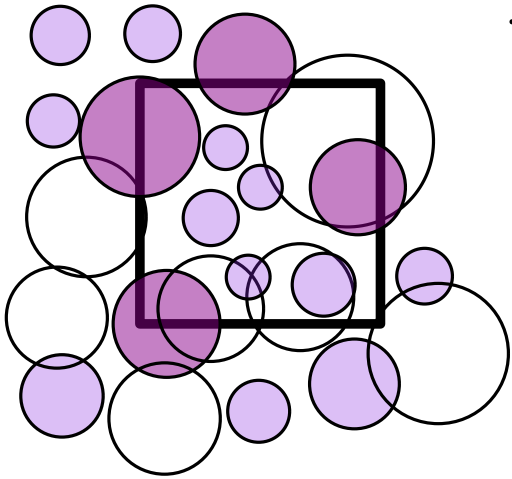


Claim. Given S, we can compute a family Y of $\mathrm{poly}(n)$ squares containing all attainable square separators of all subsets of S.

Exact algorithm for discrete packing II



for each separator $\sigma \in Y$ **do**

    for each intersecting $I_\sigma \subset S$ of size $O(\sqrt{k})$ **do**

        Remove disks in S intersecting σ

        Remove neighbors of I_σ

        Recurse on disks inside σ

        Recurse on disks outside σ

return largest indep. set found

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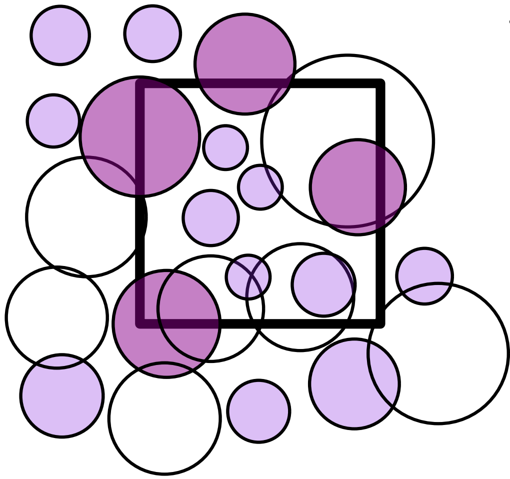

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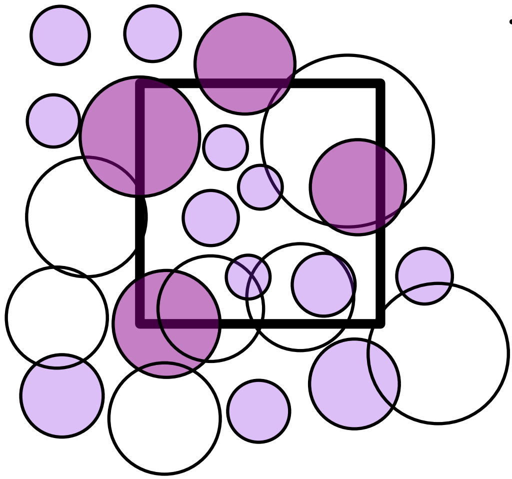

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$$T(n,k) = n^{c\sqrt{k} + c\sqrt{(36/37)k} + c\sqrt{(36/37)^2 k} + \dots} = n^{O(\sqrt{k})}$$

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General graphs: Independent set is NP-hard, has $O(n^k k^2)$ algo

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$\Downarrow$

Theorem. There is no $f(k)n^{o(\sqrt{k})}$ algorithm for Independent Set in disk graphs for any computable f, unless ETH fails.

Geometric set cover: discrete vs continuous

Set cover: given m subsets of $\{1,\ldots,n\}$, are there k among them whose union is $\{1,\ldots,n\}$

very hard, can't be approximated efficiently

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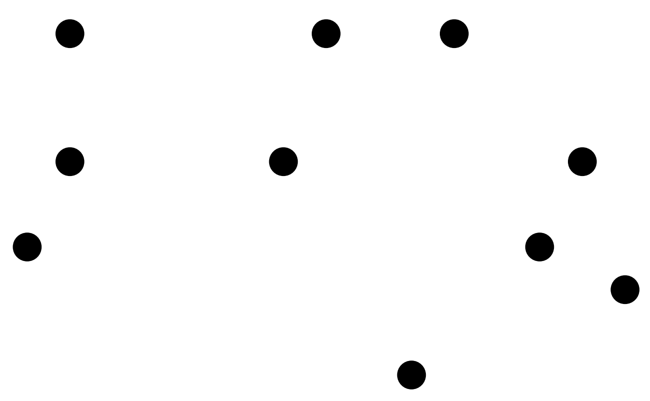
Given $P \subset \mathbb{R}^2$, can we cover P with k unit disks?



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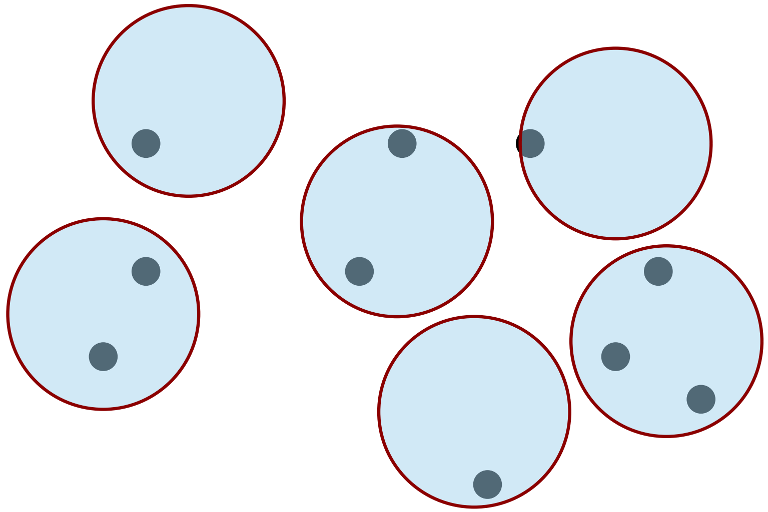

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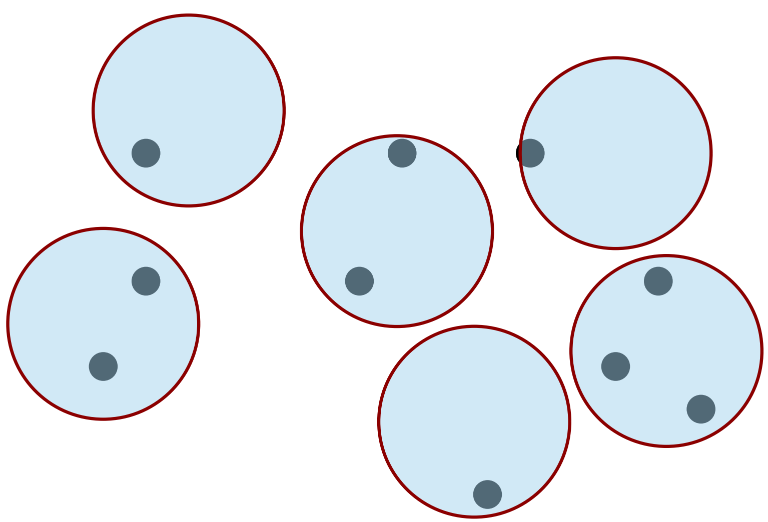

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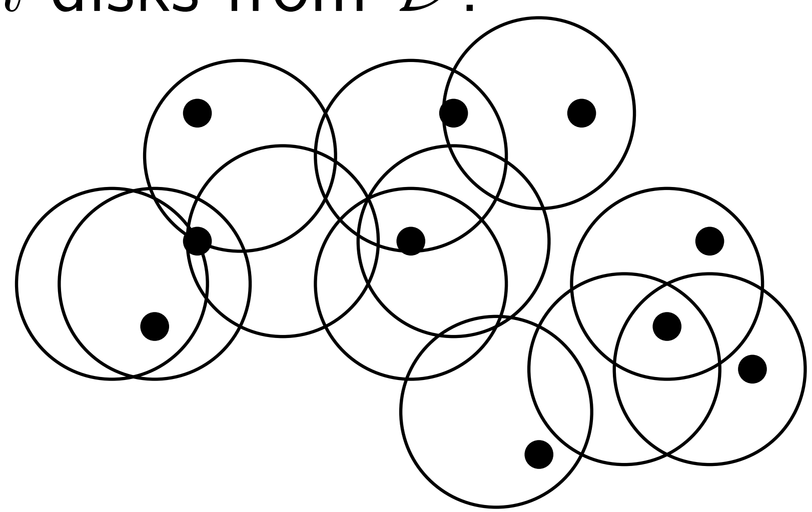
Discrete:

Given $P \subset \mathbb{R}^2$ and m unit disks $\mathcal{D}$, can we cover P with k disks from $\mathcal{D}$?



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Proof based on guessing separator in solution's Voronoi diagram.

Theorem (Marx–Pilipczuk, 2015). There is no $f(k)(m+n)^{o(\sqrt{k})}$ algorithm for covering points with disks for any computable f, unless ETH fails.

Shifting grids
Approximation schemes
Hochbaum–Maass 1985

PTASes

Definition. A polynomial time approximation scheme (PTAS) for a minimization problem is an algorithm, which given $\varepsilon > 0$ and the input instance, outputs a feasible solution of value at most $(1+\varepsilon)OPT$ in $\text{poly}_\varepsilon(n)$ time.

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Example: Independent set is APX-hard on general graphs.
But! Independent set in planar graphs has a PTAS. (Baker '83)

Packing unit disks via shifting

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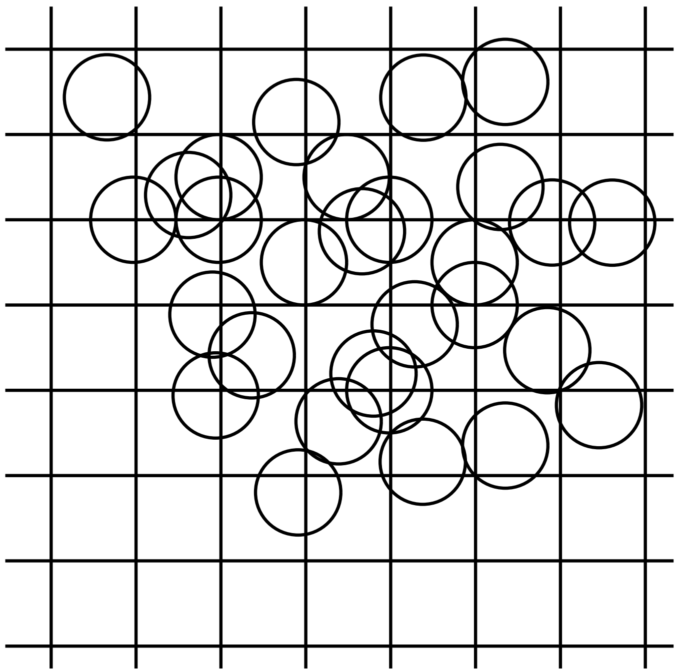

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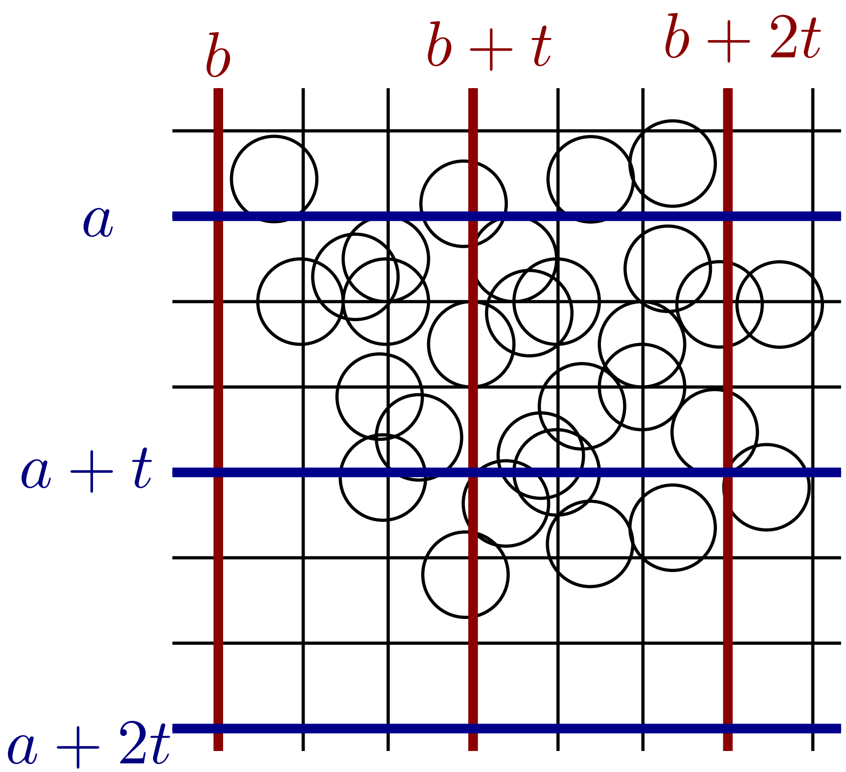

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Let $t = \lceil 2/\varepsilon \rceil$.
For a shift (a,b) ($a, b \in \{0, \dots, t-1\}$), select horizontal lines $a, a+t, a+2t, \dots$
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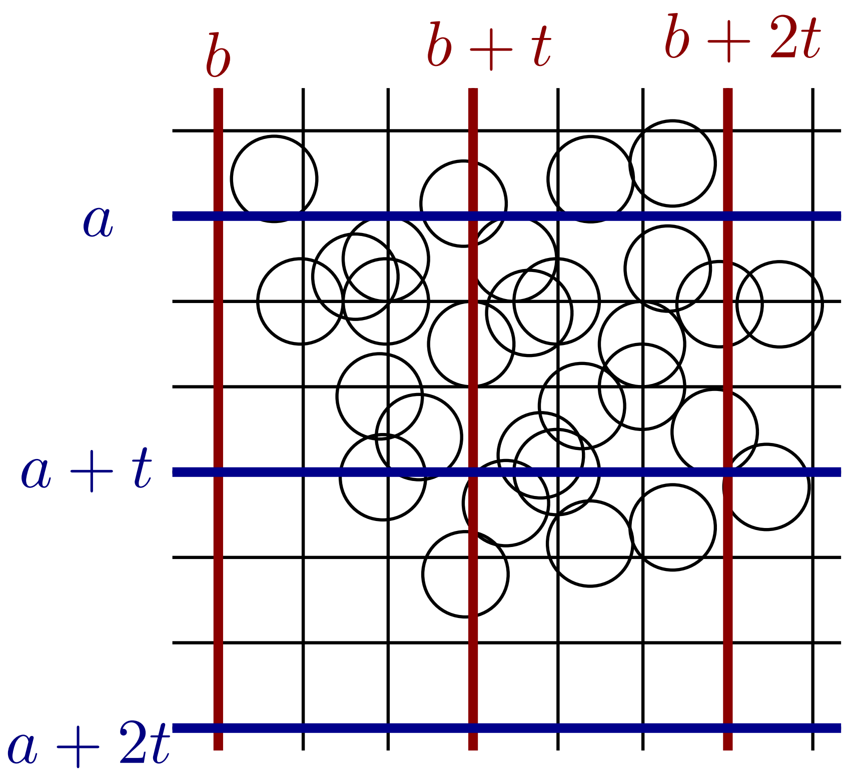

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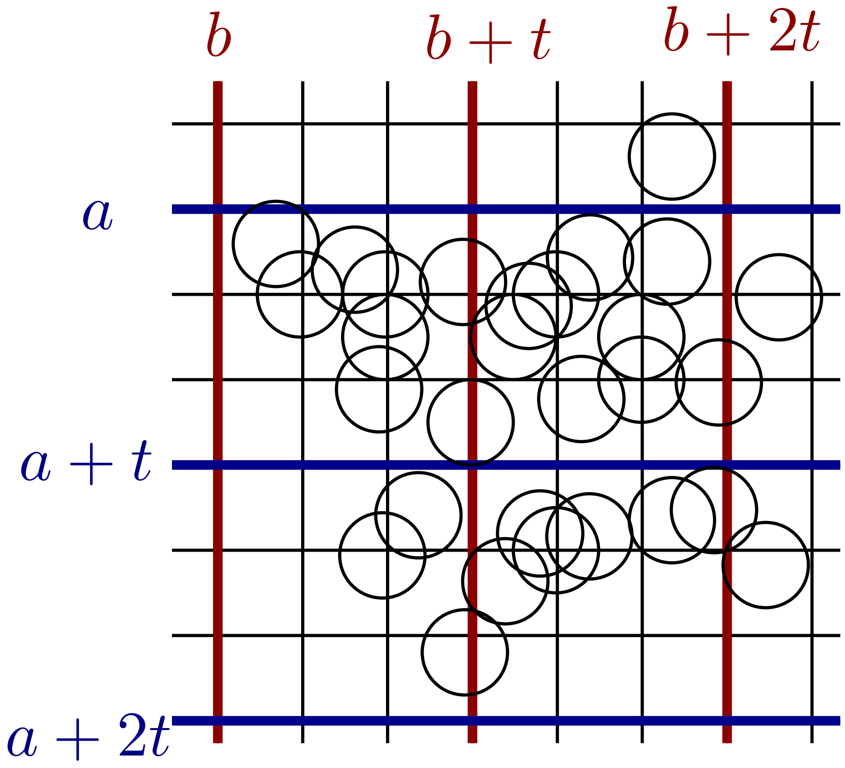
Remove disks intersecting selected lines

Shifting startegy: solving cells



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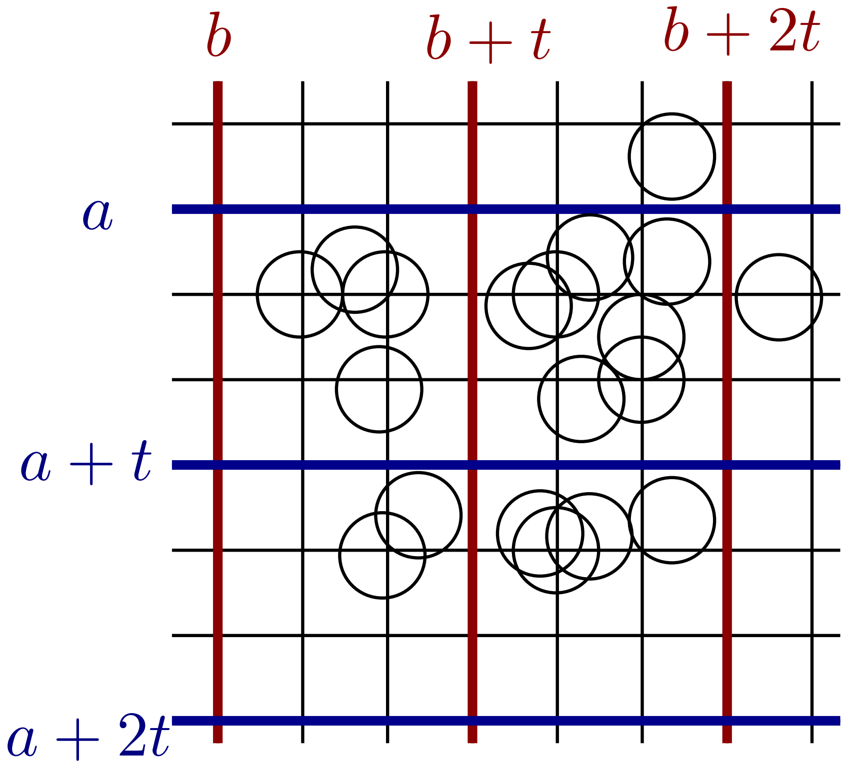

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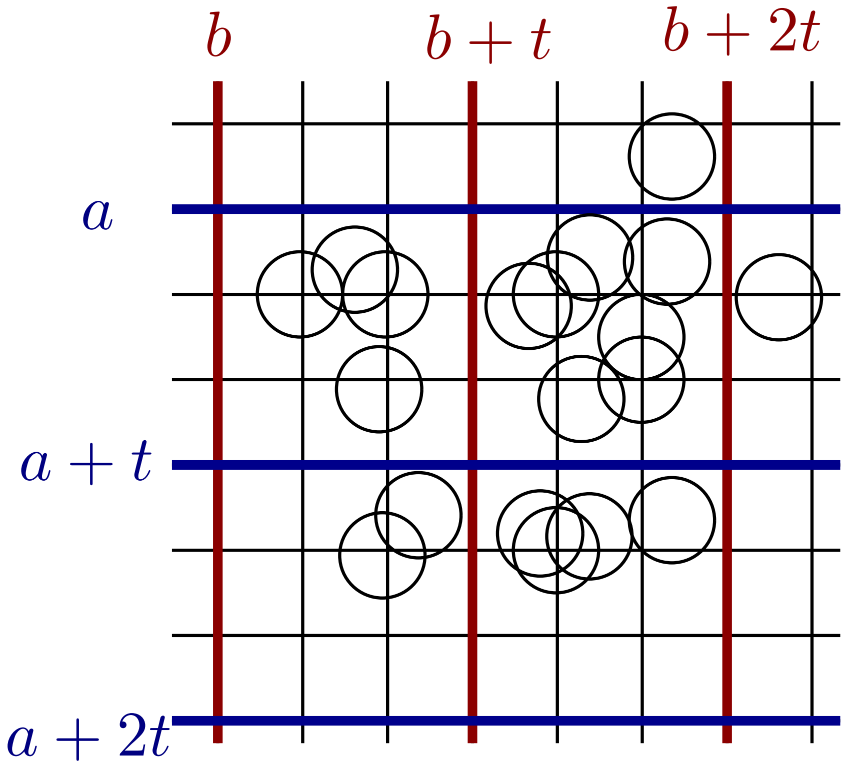

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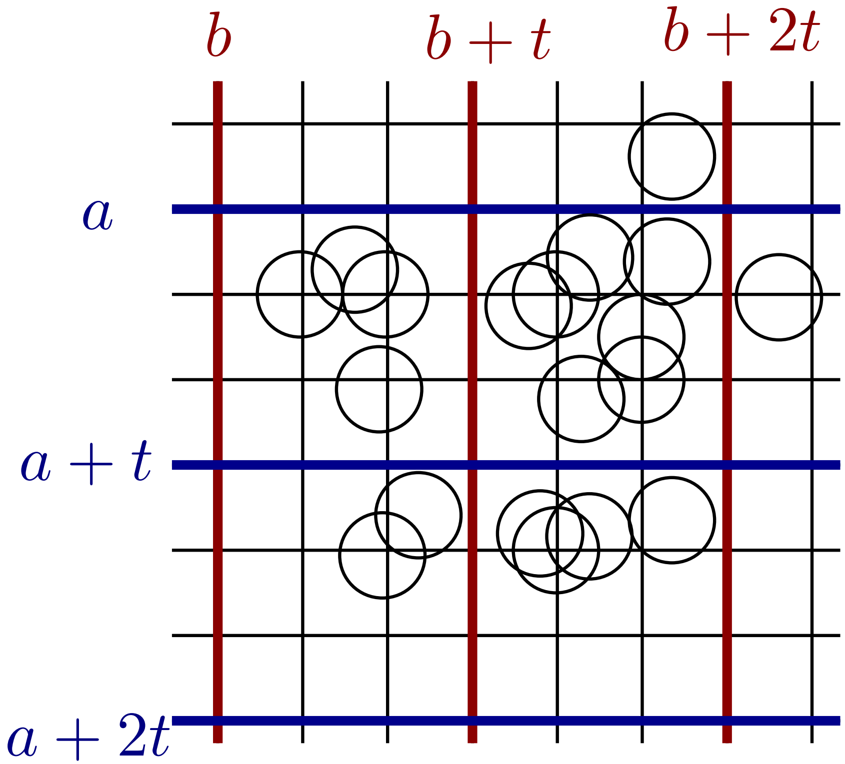

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Proof. Of the $t = \lceil 2/\varepsilon \rceil$ shifts for horizontals, there is some $a \in \{0,\dots,t-1\}$ intersecting $\leq \frac{\varepsilon}{2}OPT$ solution disks.
Similarly there is b s.t. verticals intersect $\leq \frac{\varepsilon}{2}OPT$.
$\Rightarrow (a,b)$ works. □ □

Discrete packing outlook

- Extends to unit balls in higher dimensions: $n^{O(1/\varepsilon^{d-1})}$
- $n^{O(1/\varepsilon)}$ is essentially tight in $\mathbb{R}^2$ (Marx 2007)
- Local search: slower PTAS for “pseudodisks” (last lecture?)

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Best known: $n^{O((\log\log n/\varepsilon)^4)}$ (Chuzhoy–Ene 2016)

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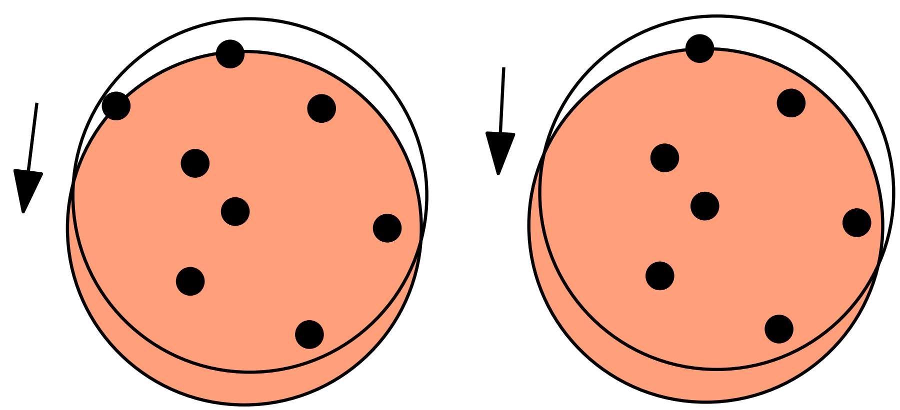
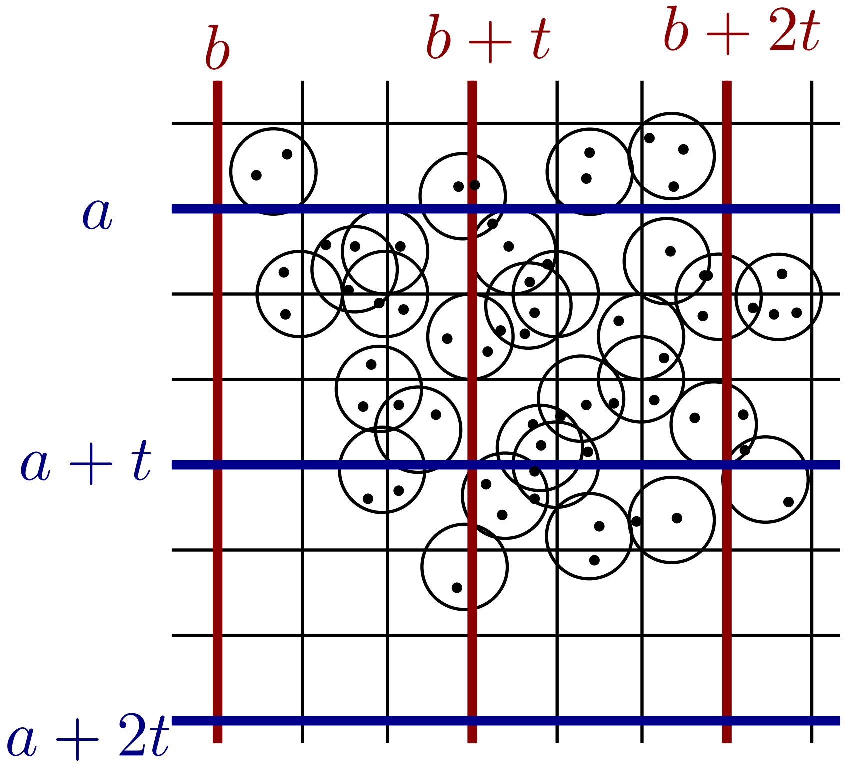
Proof. A unit disk is canonical if it has 2 input points on its boundary, or its topmost point is an input point.

There is a cover of size $k \Leftrightarrow$ there is a canonical cover of size k.

2 disks per point pair $p, p' \in P$,
one disk for each $p \in P$
$2\binom{n}{2} + n \leq n^2$ canonical disks



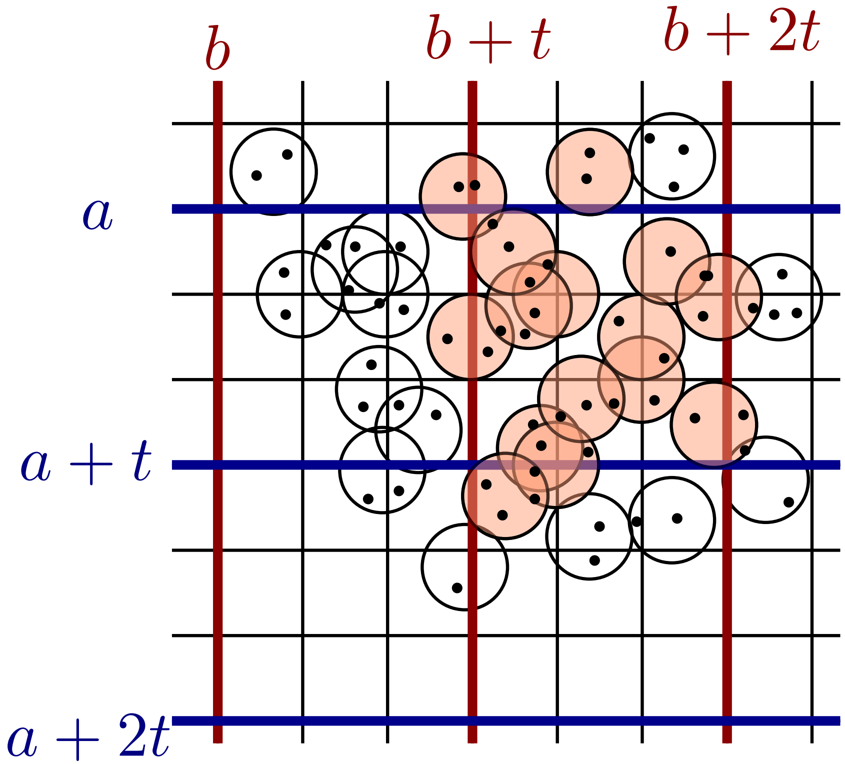
Shifting for set cover with unit disks



Grid of side length 2, set $t = \lceil 6/\varepsilon \rceil$

Cell disks: canonical disks inside and those intersecting the boundary

Shifting for set cover with unit disks



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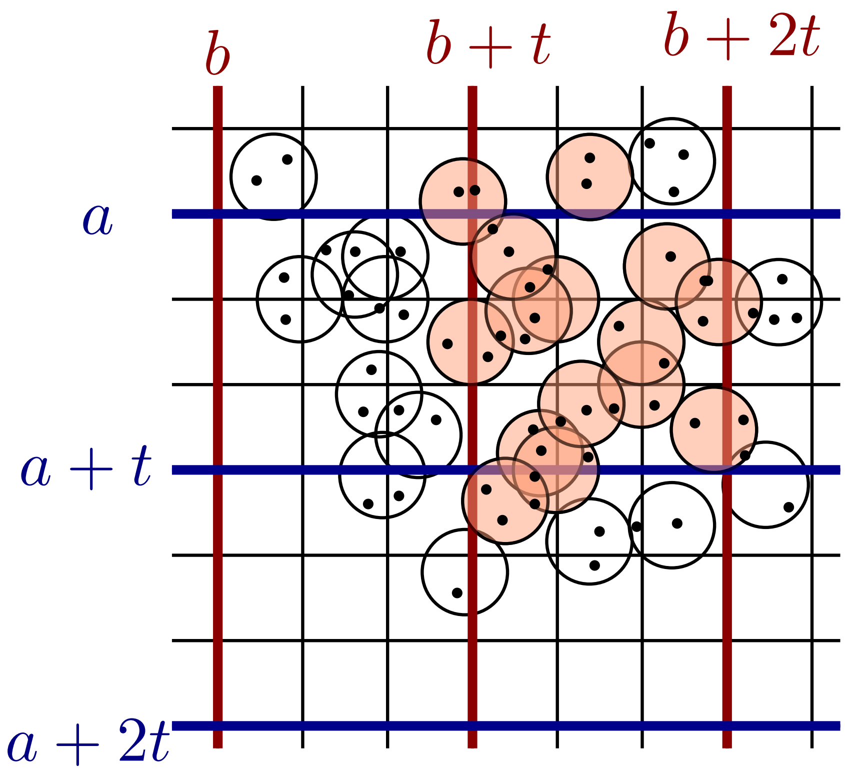

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Whole cell can be covered by $O(1/\varepsilon^2)$ (non-canoncical) disks.

$\Rightarrow$ Min cover in a cell solved in $(n^2)^{O(\sqrt{1/\varepsilon^2})} = n^{O(1/\varepsilon)}$

In C, solution $|S(C)| \leq |OPT(C)|$. Return $U := \bigcup_C S(C)$

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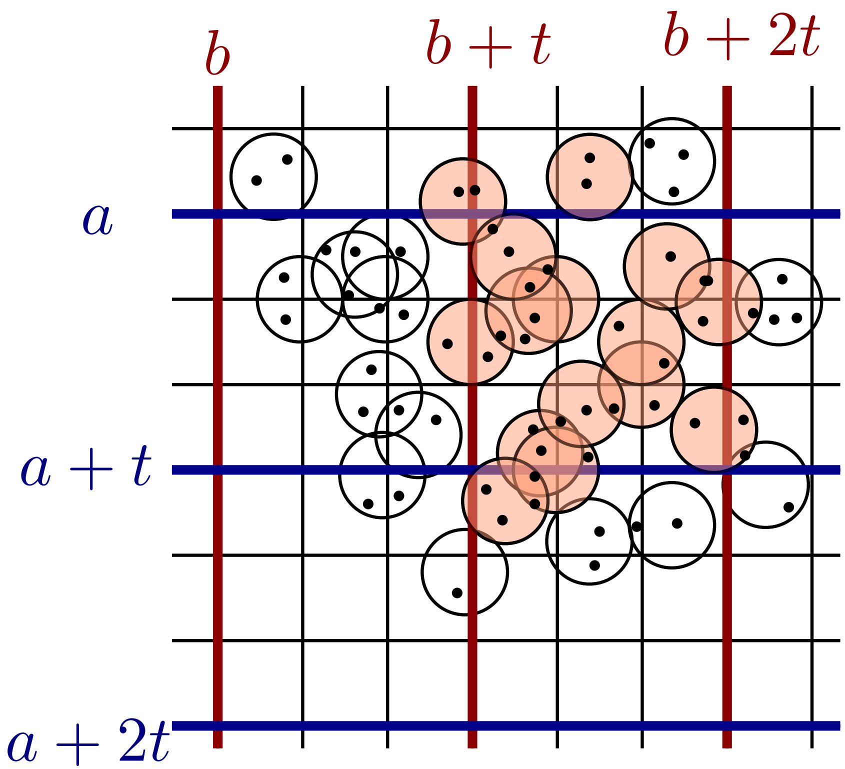

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For some shift a blue intersects $\leq |OPT|/t$ disks.

$\Rightarrow \exists (a,b)$ intersecting $2|OPT|/t \leq \varepsilon |OPT|/3$ disks.

Each disk of OPT counted in ≤ 4 cells.

$$|U| \leq \sum_C |OPT(C)| \leq |OPT| + 3\varepsilon |OPT|/3 = (1+\varepsilon)|OPT|$$