Computational Geometry, SS 2020
Assignment 2
Due: Wednesday 27.5, 8:00 (morning!) CET

Send your solutions in pdf format to:

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You can discuss about the exercises in groups, but you have to hand in a solution *in your own words.*

Problem 1. (6 points)
In this exercise, we examine the gift-wrapping step of the Preparata–Hong algorithm on some edge uv. Let v be a vertex of a convex polytope P in $\mathbb{R}^3$, and let $v_1, \ldots, v_k$ be the neighboring vertices of v in clockwise order. Let H be a plane tangent to P such that $P \cap H = vv_1$, and let u be a point in H outside the line vv_1. For a point $p \in \mathbb{R}^3$, let p' denote its projection into a plane normal to uv. Show that if for some neighbor v_i we have that

$$\sphericalangle(v_1'v'v_i') > \sphericalangle(v_1'v'v_{i-1}') \text{ and } \sphericalangle(v_1'v'v_i') > \sphericalangle(v_1'v'v_{i+1}'),$$

then the plane defined by u, v and v_i is tangent to P.

Problem 2. (8 points)
Prove that the worst-case running time of the Clarkson–Shor convex hull algorithm in $\mathbb{R}^3$ is $O(n^3)$, and show that for some point sets and for some (bad) choices of random order the algorithm needs $\Theta(n^3)$ time.

Problem 3. (10 points)
Consider the problem of finding the tangent planes to a given convex polytope $P \subset \mathbb{R}^3$ through a given line $\ell \subset \mathbb{R}^3 \setminus P$. Suppose that P contains the origin. (a) What is the dual of this problem? (b) What is the dual of the Dobkin–Kirkpatrick hierarchy discussed in Lecture 3? (c) Give an algorithm that can find the tangents to P through any given query line ℓ in $O(\log(|V(P)|))$ time, using a data structure based on either the DK hierarchy of the dual of P or the dual DK hierarchy of P. (*Hint*: you can augment the DK/dual DK hierarchy with an extra pointer at each face/vertex to make it more useful.)

Problem 4. (8 points)
Let P be a 3-dimensional polytope and let H be a plane that cuts P, i.e. it contains no vertex of P but it has at least one vertex of P on each side. Let F be the set of facets of P that intersect H and let $\mathcal{S}$ be their union. So $\mathcal{S}$ is a subset of the surface of P. In this question we are interested in the boundary structure of $\mathcal{S}$, i.e. the union of the edges of P that are incident to exactly one facet that cuts H.

1. Prove or disprove: The boundary of $\mathcal{S}$ can be empty.
2. Prove or disprove: The boundary of $\mathcal{S}$ can consist of an arbitrary number of cycles.

Problem 5. (8 points)

This question is about the complexity of higher-dimensional polytopes/convex hulls. Let $p_1, \ldots, p_d$ be d points in $\mathbb{R}^d$. They define a function that maps $x \in \mathbb{R}^d$ to the determinant

$$\Delta(x, p_1, \ldots, p_d) := \begin{vmatrix} 1 & x \\ 1 & p_1 \\ \vdots & \vdots \\ 1 & p_d \end{vmatrix}.$$

We say that these points define a hyperplane iff $\Delta(x, p_1, \ldots, p_d)$ is not a constant function. In this case $\{x \in \mathbb{R}^d | \Delta(x, p_1, \ldots, p_d) = 0\}$ is the hyperplane defined by $p_1, \ldots, p_d$ and $\{x \in \mathbb{R}^d | \Delta(x, p_1, \ldots, p_d) < 0\}$ and $\{x \in \mathbb{R}^d | \Delta(x, p_1, \ldots, p_d) > 0\}$ are the two open halfspaces separated by that hyperplane.

Let S be a finite subset of $\mathbb{R}^d$. We say that the points $p_1, \ldots, p_d \in S$ span a simplex-facet of the convex hull of S (or facet, for short) iff all *other* points q in S lie in the same open halfspace of the hyperplane spanned by $p_1, \ldots, p_d$, i.e. for all such q the sign of $\Delta(q, p_1, \ldots, p_d)$ is the same.

1. Convince yourself that this definition of facet makes sense and agrees with your intuition for the cases $d = 1, 2, 3$.

We know that in $\mathbb{R}^2$ the convex hull of a set of n points can have at most n facets and in $\mathbb{R}^3$ it can have at most $2n - 4$ facets. This exercise is about the fact that for $d > 3$ the number of facets can be superlinear.

1. Let $d = 2s$ and let us "coordinatize" d-space as $(x_1, y_1, x_2, y_2, \ldots, x_s, y_s)$. Consider a set S of $n = s \cdot k$ points that is formed by the union $\bigcup_{1 \leq i \leq s} S_i$, where each S_i consists of k points placed on the unit circle in the $x_i y_i$-plane centered at the origin so that the origin is contained in the interior of the convex polygon formed by S_i in the $x_i y_i$-plane.

   Show that the convex hull of S has k^s facets (which is $\Theta(n^{d/2})$).

   (*Hint*: Don't get lost in algebra. How often can a circle intersect a hyperplane? Also, the various $x_i y_i$ planes intersect in the origin!)

Problem 6. (10 points)

Let S be a set of n points in the plane. We want to find the smallest enclosing circle of S.

Try to adapt the SLP linear programming algorithm discussed in class to find such a circle in linear expected time.