Parameterized Algorithms using Matroids

Lecture 0: Matroid Basics

Saket Saurabh

The Institute of Mathematical Sciences, India
and University of Bergen, Norway.

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Kruskal's Greedy Algorithm for MWST

Let $G = (V, E)$ be a connected undirected graph and let $w : E \to \mathbb{R}^{\geq 0}$ be a weight function on the edges.
Kruskal's so-called greedy algorithm is as follows. The algorithm consists of selecting successively edges $e_1, e_2, \ldots, e_r$.
If edges $e_1, e_2, \ldots, e_k$ has been selected, then an edge $e \in E$ is selected so that:

1. $e \notin \{e_1, \ldots, e_k\}$ and $\{e, e_1, \ldots, e_k\}$ is a forest.
2. $w(e)$ is as small as possible among all edges e satisfying (1).

We take $e_{k+1} := e$. If no e satisfying (1) exists then $\{e_1, \ldots, e_k\}$ is a spanning tree.

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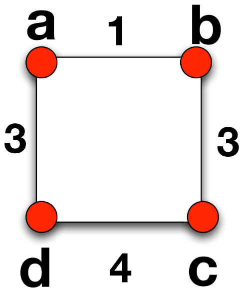
Consider Maximum Weight Matching problem.



- Application of the greedy algorithm gives *cd* and *ab*.
- However, *cd* and *ab* do not form a matching of maximum weight.

It is obviously not true that such a greedy approach would lead to an optimal solution for any combinatorial optimization problem.

It turns out that the structures for which the greedy algorithm does lead to an optimal solution, are the *matroids*.

Matroids

Definition

A pair $M = (E, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a *matroid* if it satisfies the following conditions:

(I1) $\varphi \in \mathcal{I}$ or $\mathcal{I} \neq \emptyset$.
(I2) If $A' \subseteq A$ and $A \in \mathcal{I}$ then $A' \in \mathcal{I}$.
(I3) If $A, B \in \mathcal{I}$ and $|A| < |B|$, then $\exists\ e \in (B \setminus A)$ such that $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the *hereditary property* and a pair $M = (E, \mathcal{I})$ satisfying (I1) and (I2) is called *hereditary family* or *set-family*.

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Rank and Basis

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An inclusion wise maximal set of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by $\mathsf{rank}(M)$.

Matroids and Greedy Algorithms

Let $M = (E, \mathcal{I})$ be a set family and let $w : E \to \mathbb{R}^{\geq 0}$ be a weight function on the elements.

Objective: Find a set $Y \in \mathcal{I}$ that maximizes $w(Y) = \sum_{y \in Y} w(y)$.

The *greedy algorithm* consists of selecting successively $y_1, \ldots, y_r$ as follows. If $y_1, \ldots, y_k$ has been selected, then choose $y \in E$ so that:

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We stop if no y satisfying (1) exists.

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Matroids and Greedy Algorithms

THEOREM: A set family $M = (E, \mathcal{I})$ is a matroid

if and only if

the greedy algorithm leads to a set Y in $\mathcal{I}$ of maximum weight $w(Y)$, for each weight function $w : E \to \mathbb{R}^{\geq 0}$.

Examples Of Matroids

Uniform Matroid

A pair $M = (E, \mathcal{I})$ over an n-element ground set E, is called a *uniform matroid* if the family of independent sets is given by

$$\mathcal{I} = \Big\{ A \subseteq E \mid |A| \leq k \Big\},$$

where k is some constant. This matroid is also denoted as $U_{n,k}$.

Eg: $E = \{1, 2, 3, 4, 5\}$ and $k = 2$ then

$$\begin{aligned} \mathcal{I} &= \Big\{ \{\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{1,3\}, \{1,4\}, \\ &\quad \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\} \Big\} \end{aligned}$$

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Partition Matroid

A *partition matroid* $M = (E, \mathcal{I})$ is defined by a ground set E being partitioned into (disjoint) sets $E_1, \ldots, E_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq E$ is independent if and only if $|X \cap E_i| \le k_i$ for all $i \in \{1, \ldots, \ell\}$. That is,

$$\mathcal{I} = \left\{ X \subseteq E \mid |X \cap E_i| \le k_i,\ i \in \{1, \ldots, \ell\} \right\}.$$

- If $X, Y \in \mathcal{I}$ and $|Y| > |X|$, there must exist i such that $|Y \cap E_i| > |X \cap E_i|$ and this means that adding any element e in $E_i \cap (Y \setminus X)$ to X will maintain independence.
- M in general would not be a matroid if E_i were not disjoint. Eg: $E_1 = \{1, 2\}$ and $E_2 = \{2, 3\}$ and $k_1 = 1$ and $k_2 = 1$ then both $Y = \{1, 3\}$ and $X = \{2\}$ have at most one element of each E_i but one can't find an element of Y to add to X.

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Graphic Matroid

Given a graph G, a *graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \Big\{ F \subseteq E(G) : \; F \text{ is a forest in the graph } G \Big\}$$

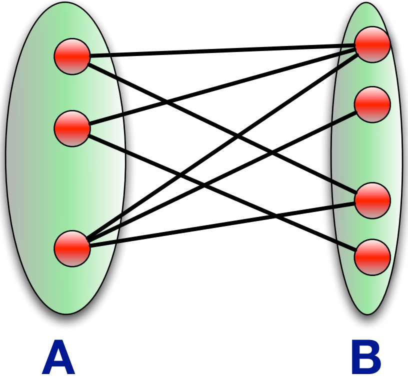
Co-Graphic Matroid

Given a graph G, a *co-graphic matroid* is defined as $M = (E, \mathcal{I})$ where and

- $E = E(G)$ – edges of G are elements of the matroid
- $$\mathcal{I} = \left\{ S \subseteq E(G) : \; G \setminus S \text{ is connected} \right\}$$

Transverse Matroid

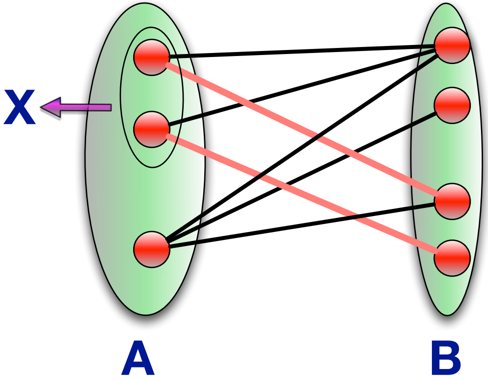
Let G be a bipartite graph with the vertex set $V(G)$ being partitioned as



Transversal Matroid

Let G be a bipartite graph with the vertex set $V(G)$ being partitioned as A and B. The *transversal matroid* $M = (E, \mathcal{I})$ of G has $E = A$ as its ground set,

$$\mathcal{I} = \left\{ X \mid X \subseteq A, \text{ there is a matching that covers } X \right\}$$



Gammoids

Let $D = (V, A)$ be a directed graph, and let $S \subseteq V$ be a subset of vertices. A subset $X \subseteq V$ is *said to be linked to* S if there are $|X|$ vertex disjoint paths going from S to X.

The paths are disjoint, not only internally disjoint. Furthermore, zero-length paths are also allowed if $X \cap S = \emptyset$.

Given a digraph $D = (V, A)$ and subsets $S \subseteq V$ and $T \subseteq V$, a *gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = T$ and

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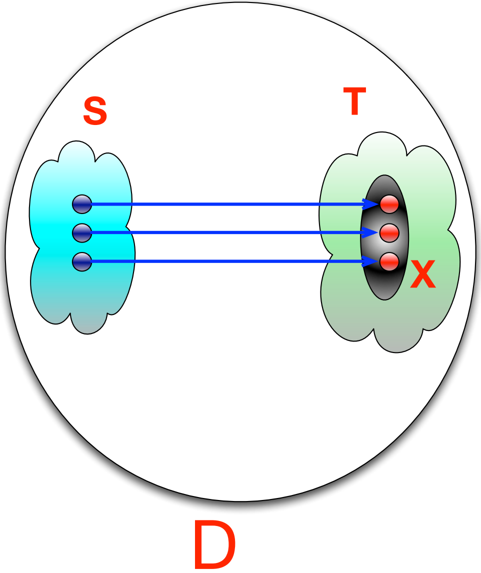
Gammoid: Example



Strict Gammoids

Given a digraph $D = (V, A)$ and subset $S \subseteq V$, a *strict gammoid* is a matroid $M = (E, \mathcal{I})$ with $E = V$ and

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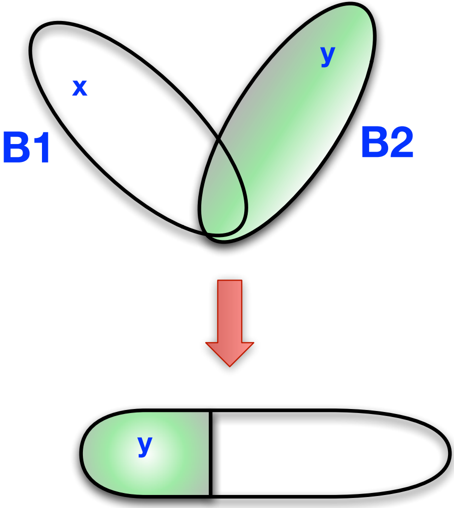
An Alternate Definition of Matroids

Basis Family

Let E be a finite set and $\mathcal{B}$ be a family of subsets of E that satisfies:

(B1) $\mathcal{B} \neq \emptyset$.
(B2) If $B_1, B_2 \in \mathcal{B}$ then $|B_1| = |B_2|$.
(B3) If $B_1, B_2 \in \mathcal{B}$ and there is an element $x \in (B_1 \setminus B_2)$ then there is an element $y \in (B_2 \setminus B_1)$ so that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.

(B3) If $B_1, B_2 \in \mathcal{B}$ and there is an element $x \in (B_1 \setminus B_2)$ then there is an element $y \in (B_2 \setminus B_1)$ so that $B_1 \setminus \{x\} \cup \{y\} \in \mathcal{B}$.



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Given $\mathcal{B}$, we define

$$\mathcal{I} = \mathcal{I}(\mathcal{B}) = \left\{ I \mid I \subseteq B \text{ for some } B \in \mathcal{B} \right\}$$

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Family of Bases

Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{B}$ be the family of bases of M – a family of maximal independent sets.
Then $\mathcal{B}$ satisfies (B1), (B2) and (B3). That is,

(B1) $\mathcal{B} \neq \emptyset$.
(B2) If $B_1, B_2 \in \mathcal{B}$ then $|B_1 = |B_2|$.
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Proof for B3

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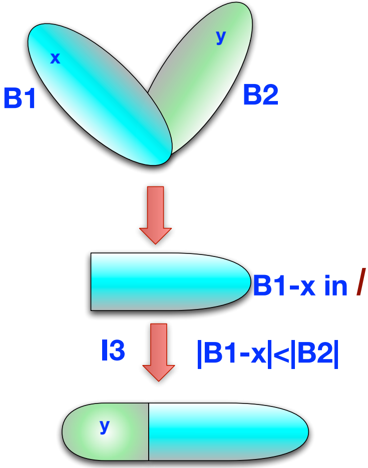

An Alternate Axiom System

THEOREM: Let E be a finite set and $\mathcal{B}$ be the family of subsets of E that satisfies (B1), (B2) and (B3) then $M = (E, \mathcal{I})$ is a matroid and $\mathcal{B}$ is the family of bases of this matroid. Recall, that

$$\mathcal{I} = \mathcal{I}(\mathcal{B}) = \left\{ I \mid I \subseteq B \text{ for some } B \in \mathcal{B} \right\}.$$

New Matroids from Old

Deletion and Contraction

Let $M = (E, \mathcal{I})$ be a matroid, and let X be a subset of E.

Deleting X from M gives a matroid $M \setminus X = (E \setminus X, \mathcal{I}')$ such that $S \subseteq E \setminus X$ is independent in $M \setminus X$ if and only if $S \in \mathcal{I}$.

$$\mathcal{I}' = \left\{ S \mid S \subseteq E \setminus X, S \in \mathcal{I} \right\}$$

Contracting X from M gives a matroid $M/X = (E \setminus X, \mathcal{I}')$ such that $S \subseteq E \setminus X$ is independent in M/X if and only if $S \cup X \in \mathcal{I}$.

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Deletion and Contraction

- **Deletion:** The bases of $M \setminus X$ are those bases of M that do not contain X.
- **Contraction:** The bases of M/X are those bases of M that do contain X with X then removed from each such basis.

Direct Sum

Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\cdots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$.

The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if for all $i \leq t$, $X \cap E_i \in \mathcal{I}_i$.

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Truncation

The *t-truncation* of a matroid $M = (E, \mathcal{I})$ is a matroid $M' = (E, \mathcal{I}')$ such that $S \subseteq E$ is independent in M' if and only if $|S| \leq t$ and S is independent in M (that is $S \in \mathcal{I}$).

Dual

Let $M = (E, \mathcal{I})$ be a matroid, $\mathcal{B}$ be the family of its bases and

$$\mathcal{B}^* = \left\{ E \setminus B \mid B \in \mathcal{B} \right\}.$$

The *dual of* a matroid M is $M^* = (E, \mathcal{I}^*)$, where

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That is, $\mathcal{B}^*$ is a family of bases of M^*.

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Matroid Representation

Remark

- Need compact representation to for the family of independent sets.
- Also to be able to test easily whether a set belongs to the family of independent sets.

Linear Matroid

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. Given A we define the matroid $M = (E, \mathcal{I})$ as follows.
A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are *linearly independent* over $\mathbb{F}$.

$$A = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} * \text{ are elements of } F$$

The matroids that can be defined by such a construction are called *linear matroids*.

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Linear Matroids and Representable Matroids

If a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is *representable* over $\mathbb{F}$.

Linear Matroids and Representable Matroids

A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^\ell$ that correspond to the elements such that the linearly independent sets of vectors precisely correspond to independent sets of the matroid.

Let $E = \{e_1, \ldots, e_m\}$ and ℓ be a positive integer.

$$\begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ \ell \end{array} & \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{\ell \times m} \end{array}$$

A matroid $M = (E, \mathcal{I})$ is called *representable* or *linear* if it is representable over some field $\mathbb{F}$.

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\end{array}
$$

A matroid $M = (E, \mathcal{I})$ is called *representable* or *linear* if it is representable over some field $\mathbb{F}$.

Linear Matroid

Let $M = (E, \mathcal{I})$ be linear matroid and Let $E = \{e_1, \ldots, e_m\}$ and $d = \mathsf{rank}(M)$.
We can always assume (using Gaussian Elimination) that the matrix has following form:

$$\left[\begin{array}{c|c} I_{d\times d} & D \end{array}\right]_{d\times m}$$

Here $I_{d\times d}$ is a $d \times d$ identity matrix.

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Direct Sum of Matroid

Let $M_1 = (E_1, \mathcal{I}_1)$, $M_2 = (E_2, \mathcal{I}_2)$, $\cdots$, $M_t = (E_t, \mathcal{I}_t)$ be t matroids with $E_i \cap E_j = \emptyset$ for all $1 \leq i \neq j \leq t$. The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (E, \mathcal{I})$ with $E := \bigcup_{i=1}^{t} E_i$ and $X \subseteq E$ is independent if and only if for all $i \leq t$, $X \cap E_i \in \mathcal{I}_i$.

Let A_i be the representation matrix of $M_i = (E_i, \mathcal{I}_i)$ over the same field $\mathbb{F}$. Then,

$$A_M = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_t \end{pmatrix}$$

is a representation matrix of $M_1 \oplus \cdots \oplus M_t$ over $\mathbb{F}$.

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Deletion of a Matroid

Let $M = (E, \mathcal{I})$ be a matroid, and let X be a subset of E. Deleting X from M gives a matroid $M \setminus X = (E \setminus X, \mathcal{I}')$ such that $S \subseteq E \setminus X$ is independent in $M \setminus X$ if and only if $S \in \mathcal{I}$.

$$\mathcal{I}' = \left\{ S \mid S \subseteq E \setminus X, S \in \mathcal{I} \right\}$$

Given a representation of A_M of M, a representation of $M \setminus X$ can be obtained by deleting the columns corresponding to X.

$$A_M = \begin{array}{c c} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ d \end{array} & \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{d \times m} \end{array}$$

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Deletion of a Matroid

Let $X = \{e_2, e_3\}$.

$$A_M = \begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ d \end{array} & \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{d\times m} \end{array}$$

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Dual of a Matroid

Let $M = (E, \mathcal{I})$ be a matroid, $\mathcal{B}$ be the family of its bases and

$$\mathcal{B}^* = \Big\{ E \setminus B \mid B \in \mathcal{B} \Big\}.$$

The *dual of* a matroid M is $M^* = (E, \mathcal{I}^*)$, where

$$\mathcal{I}^* = \Big\{ X \mid X \subseteq B,\ B \in \mathcal{B}^* \}.$$

That is, $\mathcal{B}^*$ is a family of bases of M^*.
Let $A = [I_{d \times d} \mid D]$ represent the matroid M then the matrix $A^* = [-D^T \mid I_{m-r \times m-r}]$ represents the dual matroid M^*.

Dual of a Matroid: A concrete example

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
A = & \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

$\{a, b, c, d\}$ is a basis of M then $\{e, f, g\}$ is a basis of M^*.

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\
A^* = & \begin{bmatrix} & & & & 1 & 0 & 0 \\ & & & & 0 & 1 & 0 \\ & & & & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$

To find coordinates for columns a, b, c, d, we will choose entries that make *every row of A orthogonal to every row of A^*.*

Dual of a Matroid: A concrete example

$$A = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$A^* = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

Here, D is colored in green.

Uniform Matroid

Every uniform matroid is linear and can be represented over a finite field by a $k \times n$ matrix A_M where the $A_M[i,j] = j^{i-1}$.

$$\begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ k \end{array} & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2^{k-1} & 3^{k-1} & \cdots & n^{k-1} \end{bmatrix}_{k \times n} \end{array}$$

Observe that for A_M to be representable over a finite field $\mathbb{F}$, we need that the determinant of any $k \times k$ submatrix of A_M must not vanish over $\mathbb{F}$.

The determinant of any $k \times k$ submatrix of A_M is upper bounded by $k! \times n^{k-1}$ (this follows from the Laplace expansion of determinants). Thus, choosing a field $\mathbb{F}$ of size larger than $k! \times n^{k-1}$ suffices.

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Uniform Matroid: Size of the representation

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\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 2 & 3 & \cdots & n \\
1 & 2^2 & 3^2 & \cdots & n^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
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\end{array}
$$

So the size of the representation: $O((k \log n) \times nk)$ bits.

Partition Matroid

A partition matroid $M = (E, \mathcal{I})$ is defined by a ground set E being partitioned into (disjoint) sets $E_1, \ldots, E_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq E$ is independent if and only if $|X \cap E_i| \leq k_i$ for all $i \in \{1, \ldots, \ell\}$.

Observe that a partition matroid is a direct sum of uniform matroids

$$U_{|E_1|,k_1}, \cdots, U_{|E_\ell|,k_\ell}, \quad U_{|E_1|,k_1} \oplus \cdots \oplus U_{|E_\ell|,k_\ell}$$

A uniform matroid $U_{n,k}$ is representable over a field $\mathbb{F}$ of size larger than $k! \times n^{k-1}$ and thus a partition matroid is representable.

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Graphic Matroid

The graphic matroid is representable over any field of size at least 2.

Consider the matrix A_M with a row for each vertex $i \in V(G)$ and a column for each edge $e = ij \in E(G)$. In the column corresponding to $e = ij$, all entries are 0, except for a 1 in i or j (arbitrarily) and a -1 in the other.

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & \cdots & e_m \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & -1 & \cdots & -1 \end{bmatrix}_{n \times |E(G)|} \end{array}$$

This is a representation over reals.

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\begin{array}{c}
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\begin{bmatrix}
1 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 1 \\
1^{-1} & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1^{-1} & 1^{-1} & \cdots & 1^{-1}
\end{bmatrix}_{n \times |E(G)|}
\end{array}
$$

To obtain a representation over a field $\mathbb{F}$, one simply needs to take the representation given above over reals and simply replace all -1 by the additive inverse of 1

Transversal Matroid

For the bipartite graph with partition A and B, form an incidence matrix A_M as follows. Label the rows by vertices of B and the columns by the vertices of A_M and define:

$$a_{ij} = \begin{cases} z_{ij} & \text{if there is an edge between } a_i \text{ and } b_j, \\ 0 & \text{otherwise.} \end{cases}$$

where z_{ij} are in-determinants. Think of them as independent variables.

$$T = \begin{array}{cc} & \begin{array}{cccccc} a_1 & a_2 & \cdots & a_j & \cdots & a_\ell \end{array} \\ \begin{array}{c} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{array} & \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1j} & \cdots & z_{1\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{i1} & z_{i2} & \cdots & z_{ij} & \cdots & z_{i\ell} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nj} & \cdots & z_{n\ell} \end{bmatrix} \end{array}$$

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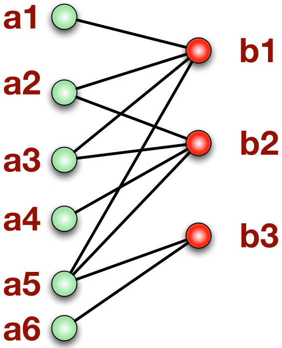
Example of the Construction



$$
\begin{array}{c c}
 & \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array} \\
\begin{array}{c} b_1 \\ b_2 \\ b_3 \end{array} & \begin{bmatrix} z_{11} & z_{12} & z_{13} & 0 & z_{15} & 0 \\ 0 & z_{22} & z_{23} & z_{24} & z_{25} & 0 \\ 0 & 0 & 0 & 0 & z_{35} & z_{36} \end{bmatrix}
\end{array}
$$

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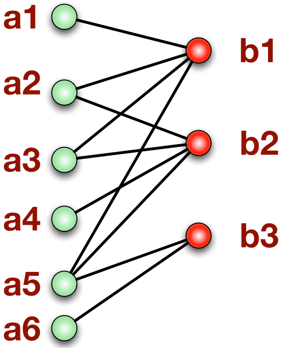

$$
\begin{array}{c}
\\ b_1 \\ b_2 \\ b_3
\end{array}
\begin{array}{c}
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0 & 0 & 0 & 0 & z_{35} & z_{36}
\end{bmatrix}
\end{array}
$$

Permutation expansion of Determinants

THEOREM: Let

$$A = (a_{ij})_{n\times n}$$

be a $n \times n$ matrix with entries in $\mathbb{F}$. Then

$$\det(A) = \sum_{\pi\in S_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} a_{i\pi(i)}.$$

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Forward direction: (Board for Picture)

- Suppose some subset $X = \{a_1, \ldots, a_q\}$ is independent.
- Then there is a matching that saturates X. Let $Y = \{b_1, b_2, \ldots, b_q\}$ be the endpoints of this matching and $a_i b_i$ are the matching edges.
- Consider $T[Y, X]$ – a submatrix with rows in Y and columns in X. Consider the determinant of $T[Y, X]$ then we have a term

$$\prod_{i=1}^{q} z_{ii}$$

which can not be cancelled by any other term! So these columns are linearly independent.

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Reverse direction: (Board for Picture)

- Suppose some subset $X = \{a_1, \ldots, a_q\}$ of columns is independent in T.
- Then there is a submatrix of $T[\star, X]$ that has non-zero determinant – say $T[Y, X]$.
- Consider the determinant of $T[Y, X]$

$$0 \neq \det(T[Y, X]) = \sum_{\pi \in S(Y)} \mathrm{sgn}(\pi) \prod_{i=1}^{q} z_{i\pi(i)}.$$

- This implies that we have a term

$$\prod_{i=1}^{q} z_{i\pi(i)} \neq 0$$

 and this gives us that there is a matching that saturates X in and hence X is independent.

Proof that Transversal Matroid is Representable over $F[\vec{z}]$

Reverse direction: (Board for Picture)

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  and this gives us that there is a matching that saturates X in and hence X is independent.
- For this direction we do not use z_{ij}, the very fact that X forms independent set of column is enough to argue that there is a matching that saturates X.

Removing z_{ij}

To remove the z_{ij} we do the following.

> *Uniformly at random assign z_{ij} from values in finite field $\mathbb{F}$ of size P.*

What should be the upper bound on P? What is the probability that the randomly obtained T is a representation matrix for the transversal matroid.

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Using Zippel-Schwartz Lemma

THEOREM: Let $p(x_1, x_2, \ldots, x_n)$ be a non-zero polynomial of degree d over some field $\mathbb{F}$ and let S be an N element subset of $\mathbb{F}$. If each x_i is independently assigned a value from S with uniform probability, then $p(x_1, x_2, \ldots, x_n) = 0$ with probability at most $\frac{d}{N}$.

- We think $\det(T[Y, X])$ as polynomial in z_{ij}'s of degree at most $n = |A|$.
- Probability that $\det(T[Y, X]) = 0$ is less than $\frac{n}{P}$. There are at most 2^n independent sets in A and thus by union bound probability that not all of them are independent in the matroid represented by T is at most $\frac{2^n n}{P}$.

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- Thus probability that T is the representation is at least $1 - \frac{2^n n}{P}$. Take P to be some field with at least $2^n n 2^n$ elements :-).
- size of this representation with be like $n^{O(1)}$ bits!

Strict Gammoids: Idea of Proof for Representation

- Show that this is a dual of a transversal matroid.
- Transversal Matroids are representable.
- Dual of a Representable Matroid is Representable.
- Conclude that Strict Gammoids are representable.

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Transversal Matroid and Strict Gammoids

Let $D = (V, A)$ be a directed graph and $S \subseteq V = \{v_1, \ldots, v_n\}$.
Let $M = (E, \mathcal{I})$ be the corresponding strict gammoid.

- For transversal matroid we need a bipartite graph. Let G^* be a bipartite graph with bipartition U and W.
  $U = \{u_i \mid v_i \in V\}$ and $W = \{w_i \mid w_i \in V \setminus S\}$.
- For each edge $v_i \in V \setminus S$ we have an edge $u_i w_i$ and for every edge $\overrightarrow{v_i v_j}$ there is an edge $u_i w_j$.

Want to show:
Let $V_0 \in \mathcal{I}$ and $|V_0| = |S|$. V_0 is linked to S if and only if there is a matching that saturates $U \setminus U_0$ in G^*.

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Proof Forward Direction

Assume first that there are $|S|$ disjoint paths going from S to V_0. Consider the matching as follows:

- $w_i \in W$ is matched to u_j if $\overrightarrow{v_j v_i}$ is an arc on some path; else
- w_i is matched to u_i.
- This means that u_i is matched if one of the paths reaches v_i and continues further on, or if none of the paths reaches v_i.
- Thus the unmatched u_is corresponds to the end points of the paths, as required.

Proof Reverse Direction

- There is a matching that saturates $U \setminus U_0$.
- Now for every vertex corresponding to S that is not in V_0 grow maximal path using matching edges.
- This corresponds to paths from S to V_0 in D.

Representation of Gammoids

- Find the representation of strict-gammoids over a large field $\mathbb{F}$.
- Delete the columns corresponding to $V \setminus T$.
- Now reduce the size of the numbers using modular arithmetic over a prime with at most $O((|S| + |T|)^{O(1)})$ bits.
- One can show: that the size of representation now will be $O((|S| + |T|)^{O(1)})$.

Final Slide

Thank You!

Any Questions?