Parameterized Algorithms using Matroids

Lecture II: Representative Sets

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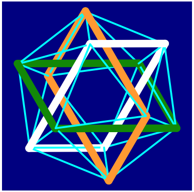


ASPAK, IMSc, March 3–8, 2014

Problems we would be interested in...

Vertex Cover

Input: A graph $G = (V, E)$ and a positive integer k.

Parameter: k

Question: Does there exist a subset $V' \subseteq V$ of size at most k such that for every edge $(u, v) \in E$ either $u \in V'$ or $v \in V'$?

Hamiltonian Path

Input: A graph $G = (V, E)$

Question: Does there exist a path P in G that spans all the vertices?

Path

Input: A graph $G = (V, E)$ and a positive integer k.

Parameter: k

Question: Does there exist a path P in G of length at least k?

REPRESENTATIVE SETS

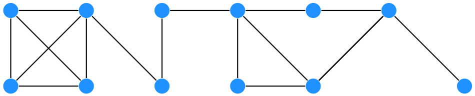
Why, What and How.

REPRESENTATIVE SETS

Why, What and How.

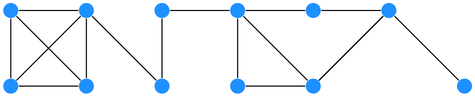
Dynamic Programming for Hamiltonian Path

◦ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad n-1 \quad n$

Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad n-1 \quad n$

v_1

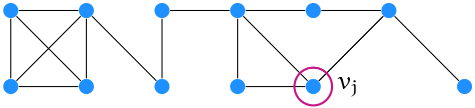
$\vdots$

v_j

$\vdots$

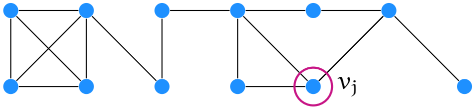
v_n

Ham-Path



	1	2	3	$\cdots$	i	$\cdots$	$n-1$	n
v_1								
$\vdots$								
v_j								
$\vdots$								
v_n								

Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

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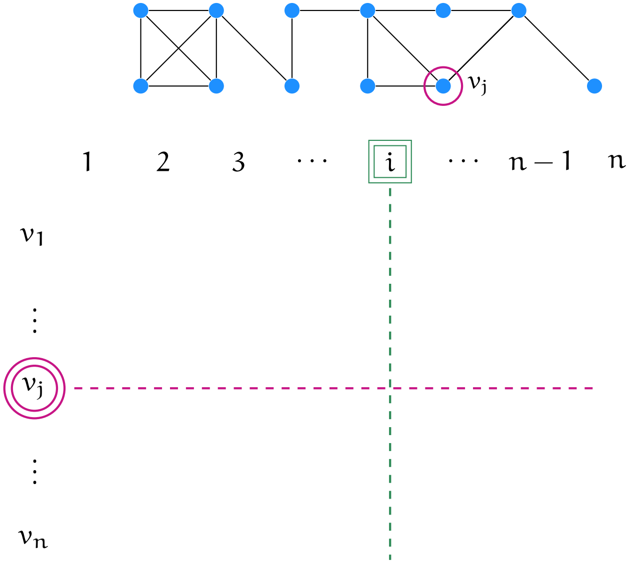
∘ Ham-Path



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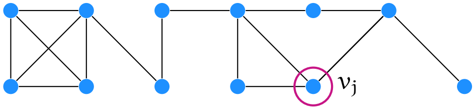

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v_j

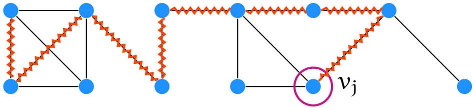
$\vdots$

v_n

V[Paths of length i ending at v_j]

∘ Ham-Path

Example:



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v_1

$\vdots$

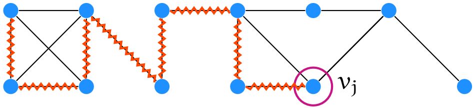
v_j

V[Paths of length i ending at v_j]

$\vdots$

v_n

Ham-Path

Example: 

$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

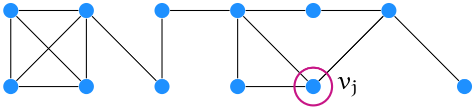
$\vdots$

v_j V[Paths of length i ending at v_j]

$\vdots$

v_n

Ham-Path



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v_j

$\vdots$

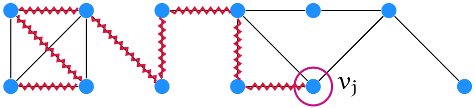
v_n

SETS, NOT SEQUENCES.

V[Paths of length i ending at v_j]

HAM-PATH

Example:



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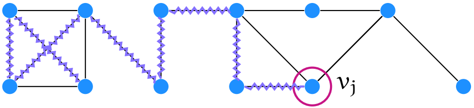
Example: 

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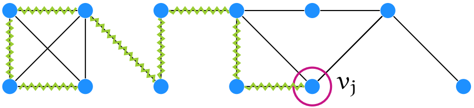

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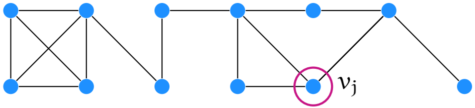

$$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$$

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$\vdots$

v_j

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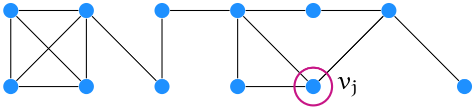
v_n

SETS, NOT SEQUENCES.

V[Paths of length i ending at v_j]

Two paths that use the same set of vertices but visit them in different orders are equivalent.

Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

v_j

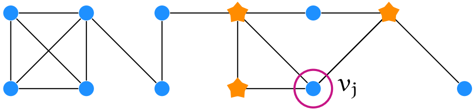
V[Paths of length i ending at v_j]

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

$\vdots$

v_n

∘ Ham-Path



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

v_j

$\vdots$

v_n

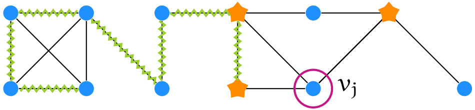
V[Paths of length i ending at v_j]

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

$u \in N(v_j)$

∘ HAM-PATH

Valid:



1   2   3 $\cdots$ $\boxed{i}$ $\cdots$ $n-1$ n

v_1

$\vdots$

v_j

$\vdots$

v_n

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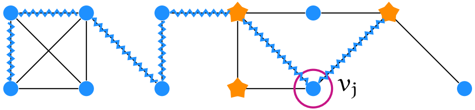
Invalid:



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

v_j

$\vdots$

v_n

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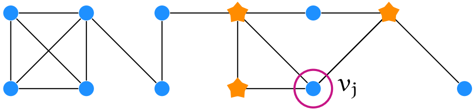
HAM-PATH



$1 \quad 2 \quad 3 \quad \cdots \quad \boxed{i} \quad \cdots \quad n-1 \quad n$

v_1

$\vdots$

Potentially storing $\binom{n}{i}$ sets.

v_j

V[Paths of length i ending at v_j]

$\vdots$

= V[Paths of length $(i-1)$ ending at u, avoiding v_j.]

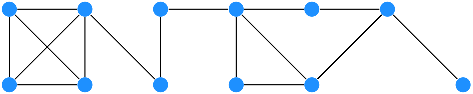
$u \in N(v_j)$

v_n

Let us now turn to k-Path.

To find paths of length at least k,
we may simply use the DP table for Hamiltonian Path
restricted to the first k columns.

K-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

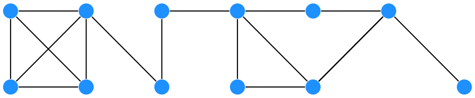
v_j

Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$

$\vdots$

v_n

K-Path



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

$\vdots$

v_j

Worst case running time: $\mathcal{O}^\star(n^k)$

$\vdots$

v_n

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In the i^{th} column, we are storing paths of length i.

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Let P be a path of length k.

There may be several paths of length i that "latch on" to the last $(k - i)$ vertices of P.

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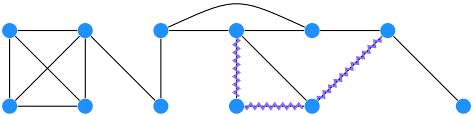
In the i^{th} column, we are storing paths of length i.

Let P be a path of length k.

There may be several paths of length i that "latch on" to the last $(k-i)$ vertices of P.

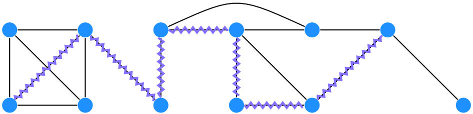
We need to store just one of them.

Example.



Example.

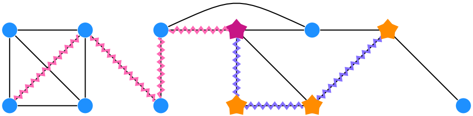
Suppose we have a path P on seven edges.

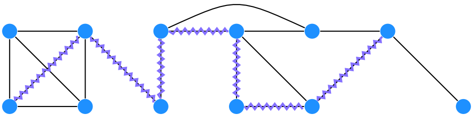


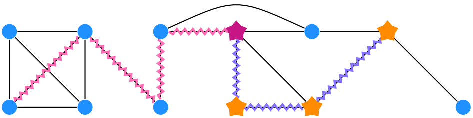
Example.

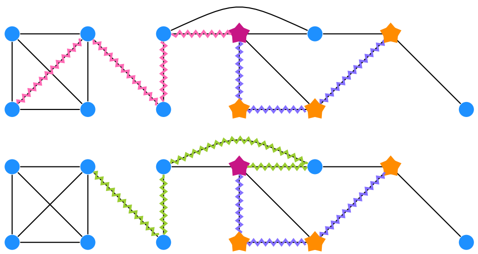
Suppose we have a path $\mathbb{P}$ on seven edges.

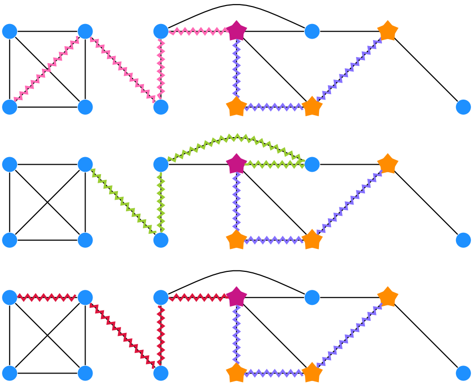
Consider it broken up into the first four and the last three edges.

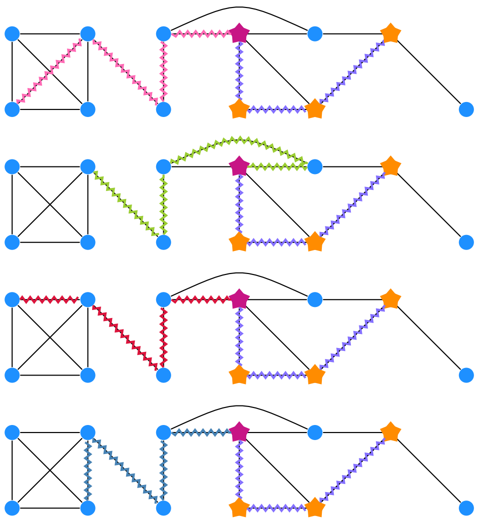


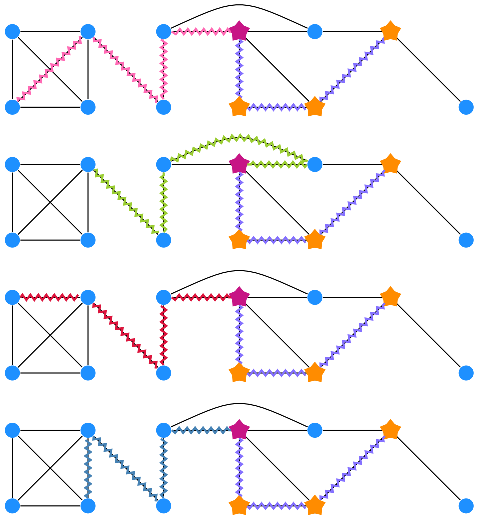






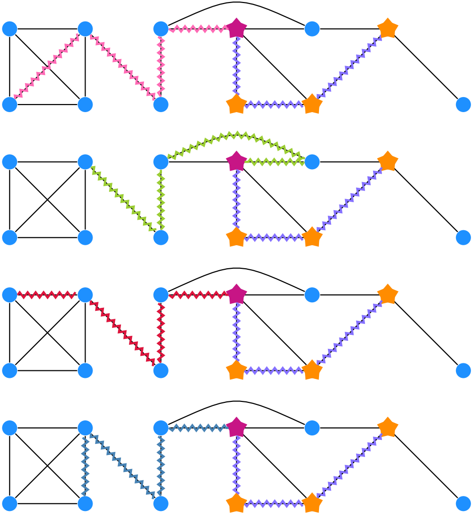






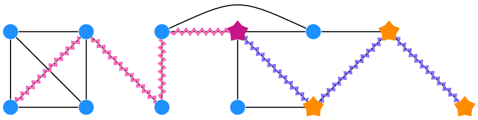
A Fixed Future ($v_{i+1} - \cdots - v_k$).

The Possibilities for Partial Solutions Compatible with $v_{i+1} - \cdots - v_k$.



A Fixed Future ($v_{i+1} - \cdots - v_k$).

Let’s try a different example.



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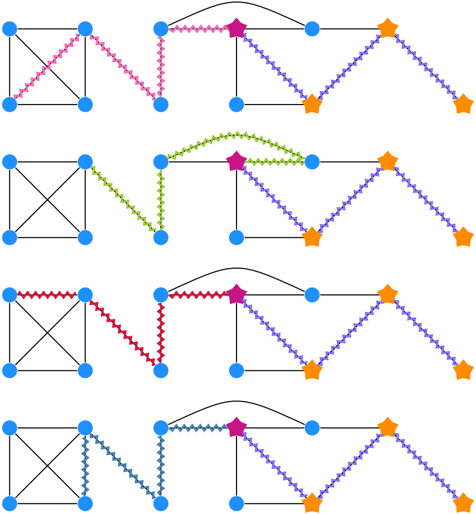
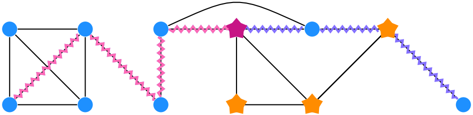
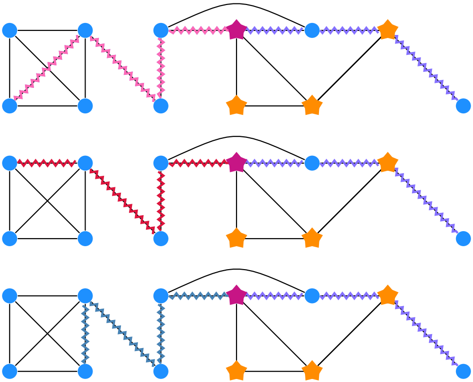


A Fixed Future ($v_{i+1} - \cdots - v_k$).

Here's one more example:



The Possibilities for Partial Solutions Compatible with $v_{i+1} - \cdots - v_k$.



A Fixed Future ($v_{i+1} - \cdots - v_k$).

For any possible ending of length $(k - i)$, we want to be sure that we store at least one among the possibly many “prefixes”.

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This could also be $\binom{n}{k-i}$.

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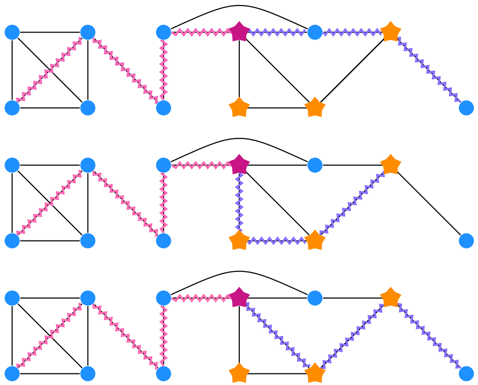
This could also be $\binom{n}{k-i}$.

The hope for “saving” comes from the fact that a single path of length i is potentially capable of being a prefix to several distinct endings.

For example...



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Partial solutions: paths of length j ending at v_i

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A “small” representative family.

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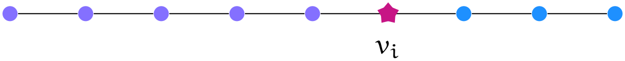

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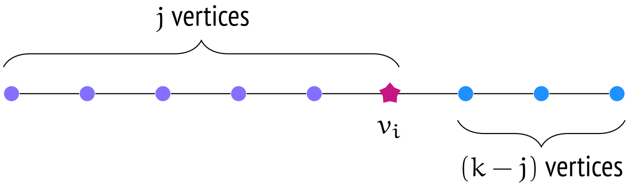

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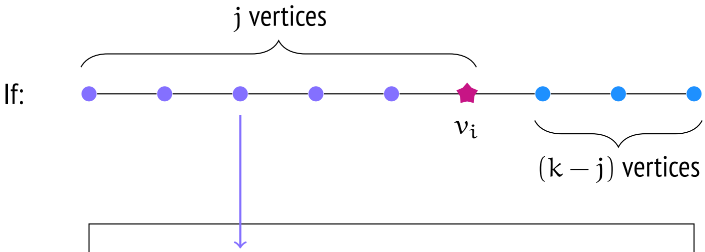

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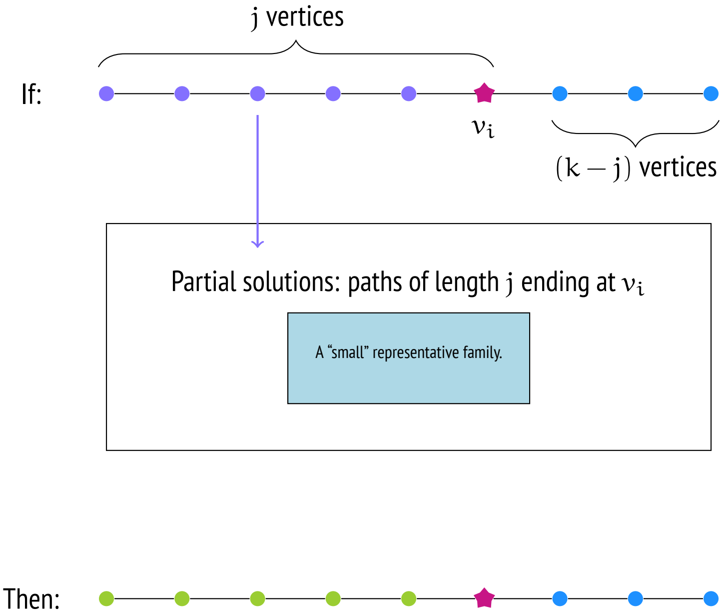
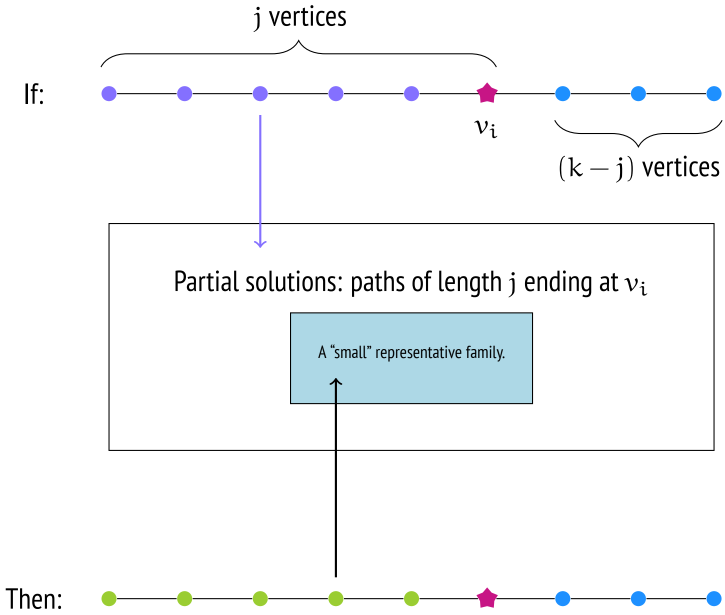

j vertices
If:
v_i
$(k-j)$ vertices
Partial solutions: paths of length j ending at v_i
A "small" representative family.
Then: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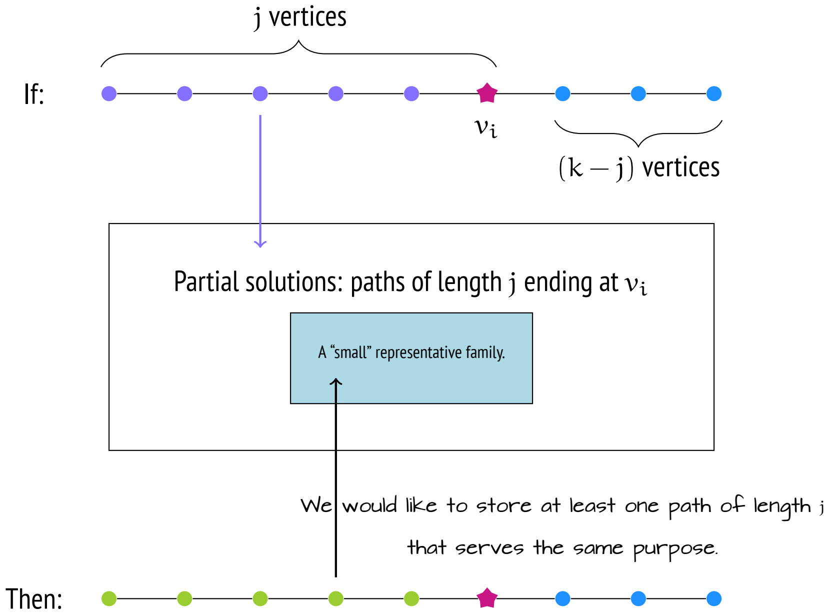

j vertices
If:
v_i
$(k - j)$ vertices
Partial solutions: paths of length j ending at v_i
A "small" representative family.
We would like to store at least one path of length j
that serves the same purpose.
Then:

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

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Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

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The "second half" of a solution – can be any subset.

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This is a valid patch into X.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

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This is a guaranteed replacement for S.

Given: A (BIG) family $\mathcal{F}$ of p-sized subsets of $[n]$.

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For any $X \subseteq [n]$ of size $(k-p)$,

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then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

Given: A $\leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

Want: A (small) subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \varnothing$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \varnothing$.

Given: A $\leqslant \binom{n}{p}$ family $\mathcal{F}$ of p-sized subsets of $[n]$.

$$S_1, S_2, \ldots, S_t$$

Known: $\exists \binom{k}{p}$ subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size $(k-p)$,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$.

Bolobás, 1965.

Given: A a matroid $(\mathcal{M}, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

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Want: A subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Given: A a matroid $(\mathcal{M}, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

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There is a subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

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Lovász, 1977

Given: A a matroid $(M, \mathcal{I})$, and a family of p-sized subsets from $\mathcal{I}$:

$$S_1, S_2, \ldots, S_t$$

There is an efficiently computable subfamily $\widehat{\mathcal{F}}$ of $\mathcal{F}$ of size at most $\binom{p+q}{p}$ such that:

For any $X \subseteq [n]$ of size at most q,

if there is a set S in $\mathcal{F}$ such that $X \cap S = \emptyset$ and $X \cup S \in \mathcal{I}$,
then there is a set $\widehat{S}$ in $\widehat{\mathcal{F}}$ such that $X \cap \widehat{S} = \emptyset$ and $X \cup \widehat{S} \in \mathcal{I}$.

Márx (2009) and Fomin, Lokshtanov, Saurabh (2013)

Summary.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

a X is disjoint from S, and

b X and S together form an independent set,

then a q-representative family $\widehat{\mathcal{F}}$ contains a set $\widehat{S}$ that is:

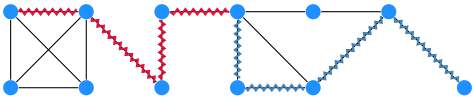
a disjoint from X, and

b forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

REPRESENTATIVE SETS

Back to Why.



$1 \quad 2 \quad 3 \quad \cdots \quad i \quad \cdots \quad k-1 \quad k$

v_1

[RECALL]

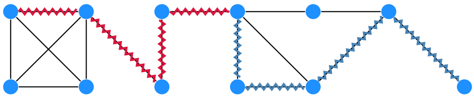
$\vdots$

Worst case running time: $\mathcal{O}^\star\left(\binom{n}{k}\right)$

v_j

$\vdots$

v_n



1 2 3 $\cdots$ i $\cdots$ $k-1$ k

v_1

[RECALL]

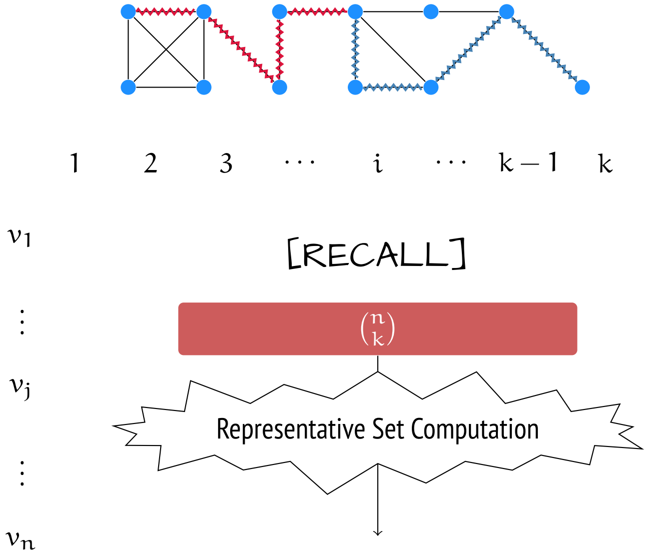
$\vdots$

$\binom{n}{k}$

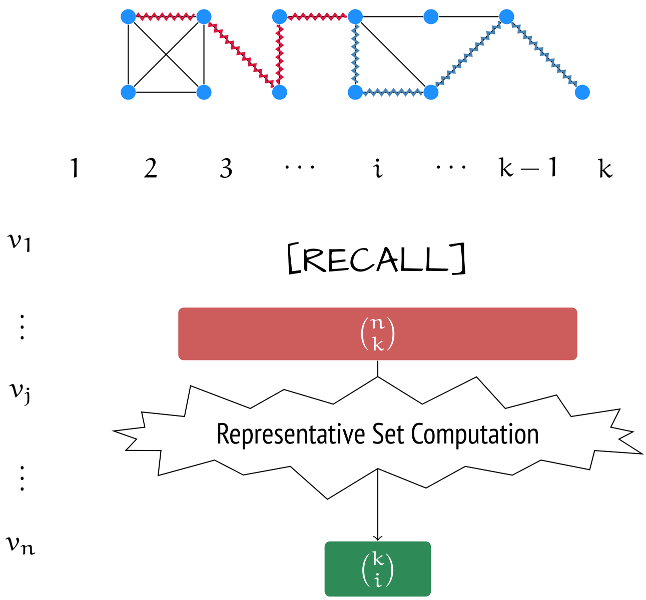
v_j

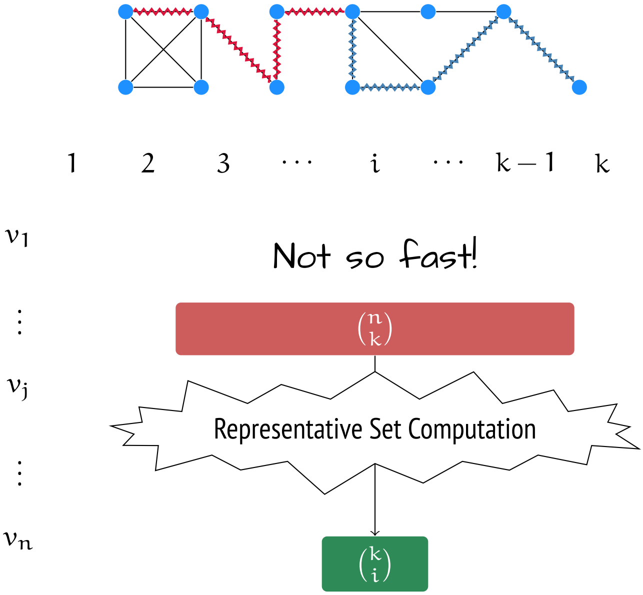
$\vdots$

v_n

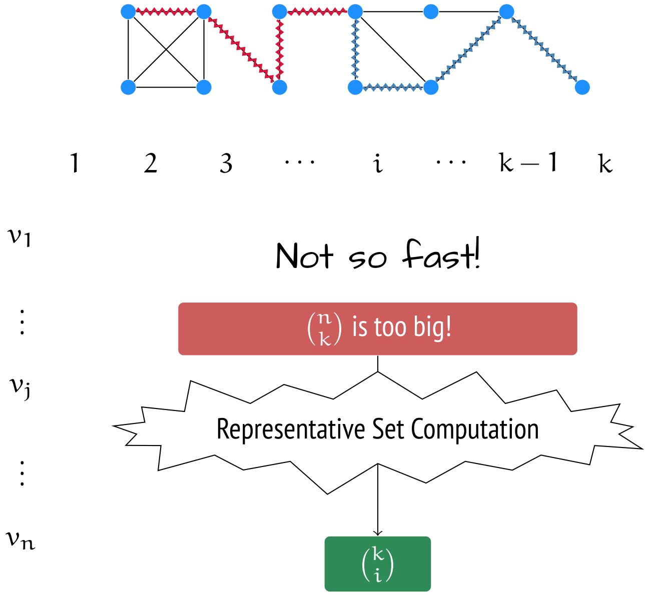


1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
⋮
v_n
[RECALL]
$\binom{n}{k}$
Representative Set Computation


1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
[RECALL]
$\binom{n}{k}$
Representative Set Computation
$\binom{k}{i}$



1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
Not so fast!
$\binom{n}{k}$
Representative Set Computation
$\binom{k}{i}$



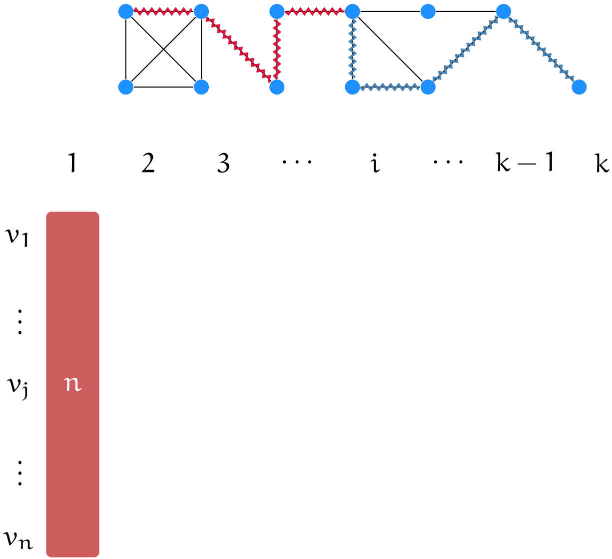
1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
⋮
v_n
Not so fast!
$\binom{n}{k}$ is too big!
Representative Set Computation
$\binom{k}{i}$

We are going to compute representative families at every intermediate stage of the computation.

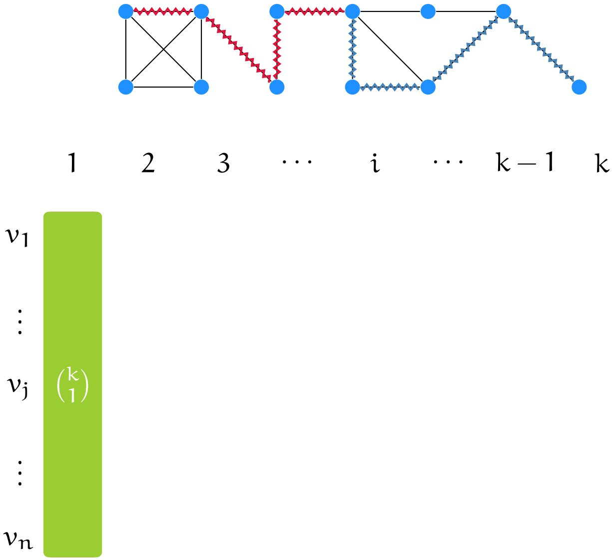
We are going to compute representative families at every intermediate stage of the computation.

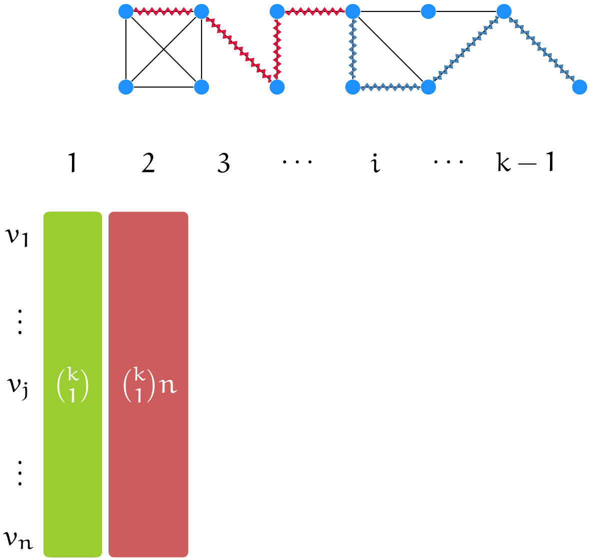
For instance, in the i^{th} column, we are storing i-uniform families.
Before moving on to column $(i+1)$, we compute $(k-i)$-representative families.

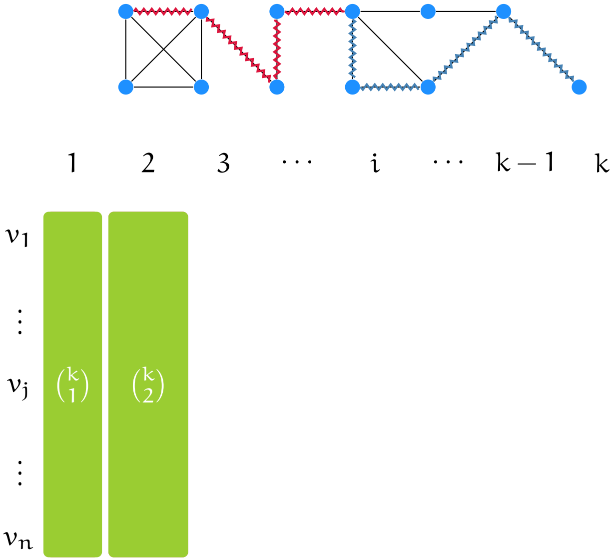
This keeps the sizes small as we go along.

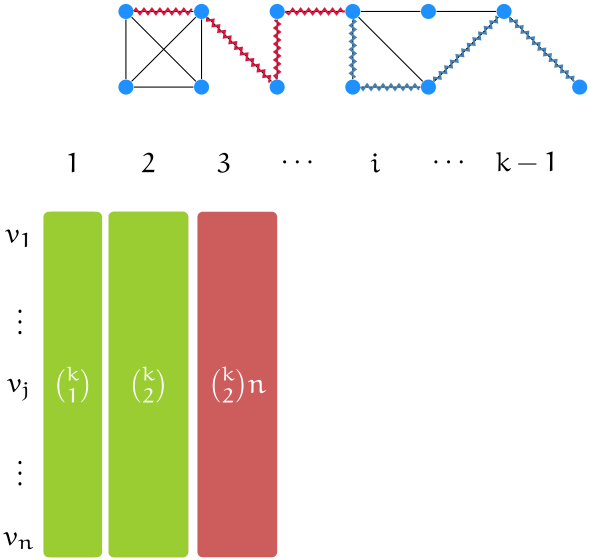


1
2
3
⋯
i
⋯
k − 1
k
v_1
⋮
v_j
n
⋮
v_n

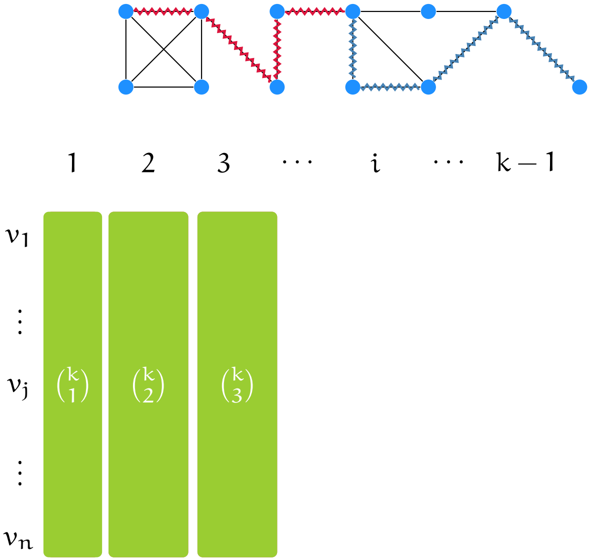

1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$


1
2
3
···
i
···
k − 1
k
v1
⋮
vj
⋮
vn
(k 1)
(k 1)n

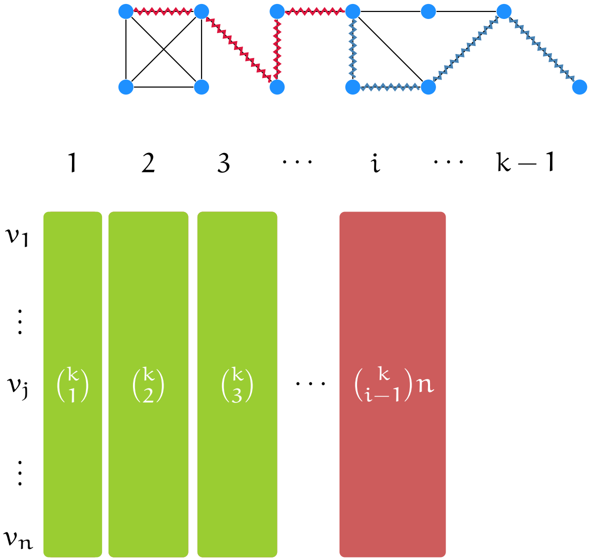

1
2
3
⋯
i
⋯
k − 1
k
v_1
v_j
v_n
$\binom{k}{1}$
$\binom{k}{2}$

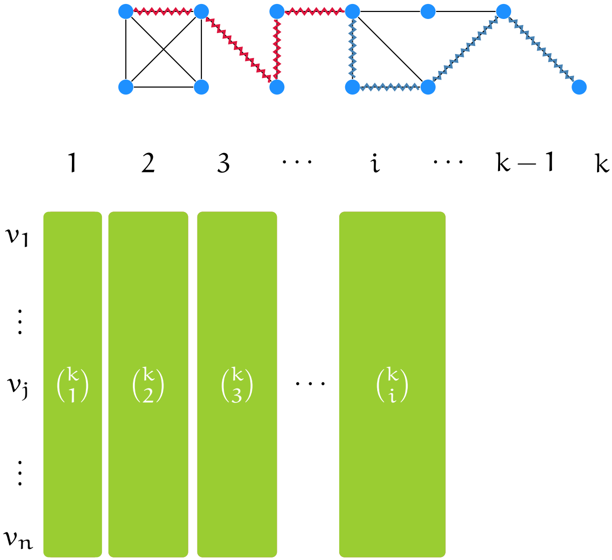


1 2 3 ⋯ i ⋯ k − 1 k
v_1
v_j
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{2}n$

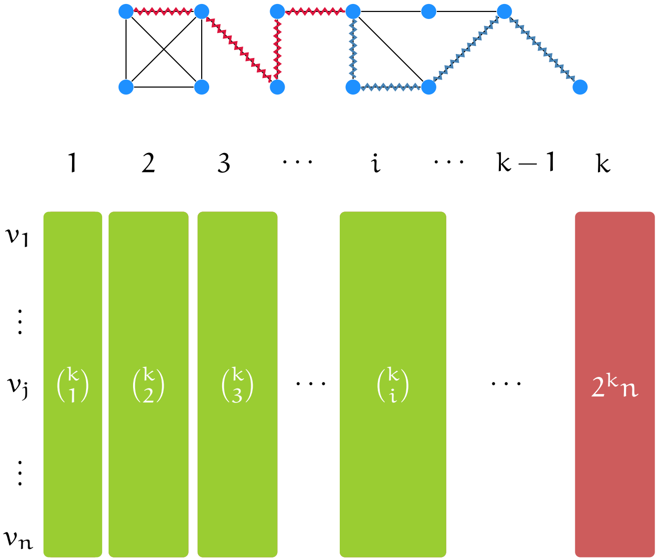


1
2
3
· · ·
i
· · ·
k − 1
k
v1
vj
vn
(k 1)
(k 2)
(k 3)

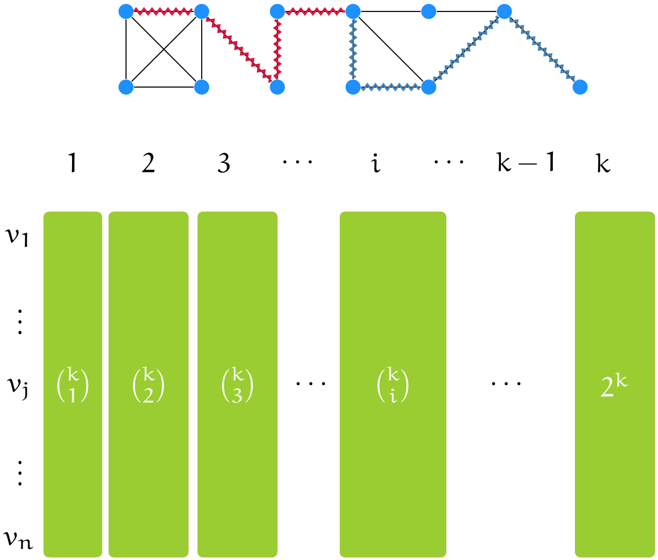

1
2
3
···
i
···
k − 1
k
v_1
⋮
v_j
⋮
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
···
$\binom{k}{i-1}n$



1
2
3
···
i
···
k − 1
k
v1
vj
vn
(k 1)
(k 2)
(k 3)
···
(k i)



1
2
3
$\cdots$
i
$\cdots$
$k-1$
k
v_1
$\vdots$
v_j
$\vdots$
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
$\cdots$
$\binom{k}{i}$
$\cdots$
$2^k n$



1
2
3
···
i
···
k − 1
k
v_1
v_j
v_n
$\binom{k}{1}$
$\binom{k}{2}$
$\binom{k}{3}$
$\binom{k}{i}$
2^k

Let $\mathcal{P}_i^j$ be the set of all paths of length i ending at v_j.

It can be shown that the families thus computed at the i^{th} column, j^{th} row are indeed $(k-i)$-representative families for P_i^j.

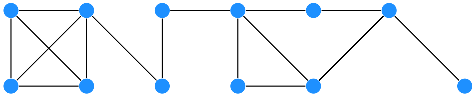
The correctness is implicit in the notion of a representative family.

REPRESENTATIVE SETS

A Different Why.

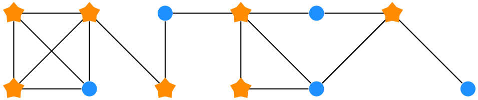
Vertex Cover

Can you delete k vertices to kill all edges?



Vertex Cover

Can you delete k vertices to kill all edges?



Let $(G = (V, E), k)$ be an instance of Vertex Cover.

Note that E can be thought of as a 2-uniform family over the ground set V.

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Note that E can be thought of as a 2-uniform family over the ground set V.

Goal: Kernelization.

In this context, we are asking if there is a small subset X of the edges such that

$G[X]$ is a YES-instance $\leftrightarrow$ G is a YES-instance.

Note: If G is a YES-instance, then $G[X]$ is a YES-instance for any subset $X \subseteq E$.

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We get one direction for free!

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It is the NO-instances that we have to worry about preserving.

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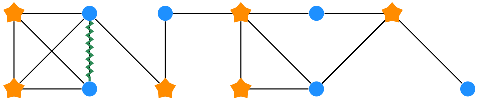
We get one direction for free!

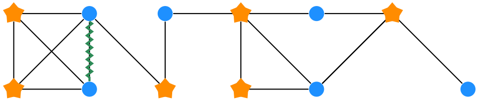
It is the NO-instances that we have to worry about preserving.

What is a NO-instance?



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S.



If G is a NO-instance:

For any subset S of size at most k,
there is an edge that is disjoint from S.

Ring a bell?

Recall.

We have at hand a p-uniform collection of independent sets, $\mathcal{F}$ and a number q. Let X be any set of size at most q. For any set $S \in \mathcal{F}$, if:

- a X is disjoint from S, and
- b X and S together form an independent set,

then a q-representative family contains a set $\widehat{S}$ that is:

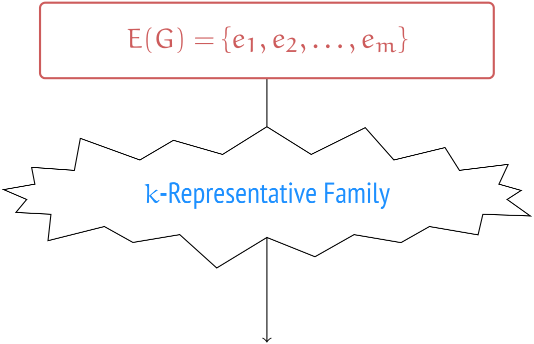
- a disjoint from X, and
- b forms an independent set together with X.

Such a subfamily is called a q-representative family for the given family.

Claim: A k-representative family for E is in fact an $O(k^2)$ kernel for vertex cover.

$$E(G) = \{e_1, e_2, \ldots, e_m\}$$

Is there a Vertex Cover of size at most k?



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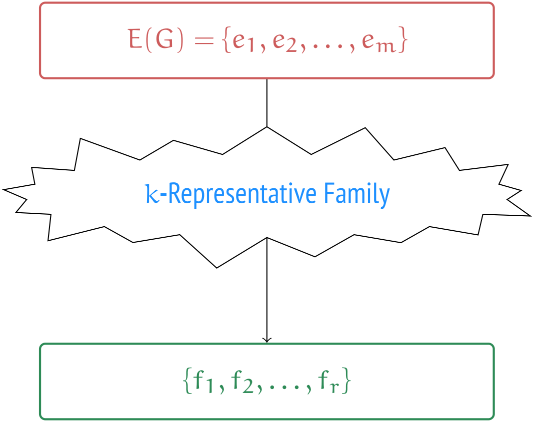

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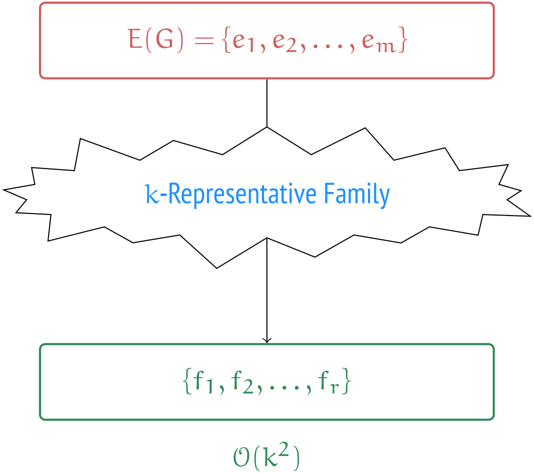

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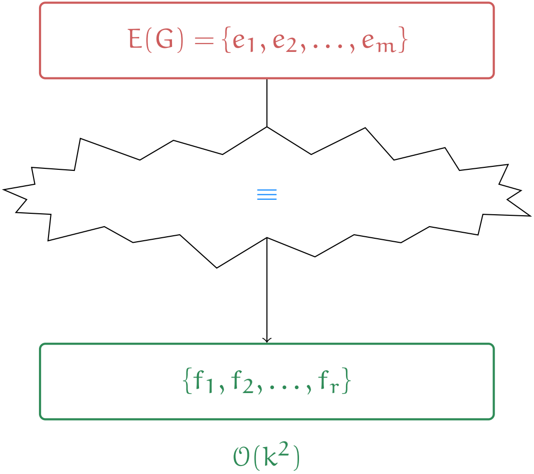

Is there a Vertex Cover of size at most k?

Let us show that if $G[X]$ is a YES-instance, then so is G.

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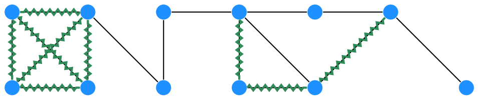
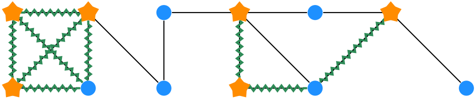
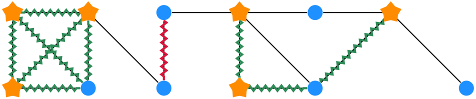
This time, by contradiction.

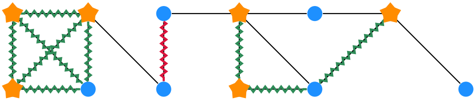




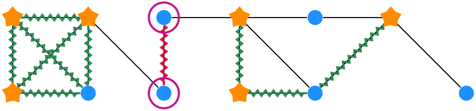
Try the solution for $G[X]$ on G.



Suppose there is an uncovered edge.

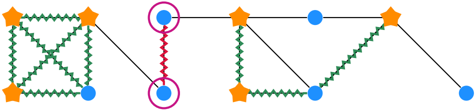


Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:



Since X is a k-representative family, for ANY $S \subseteq V$, where $|S| \leqslant k$:

if there is a set e in E such that $e \cap S = \emptyset$,



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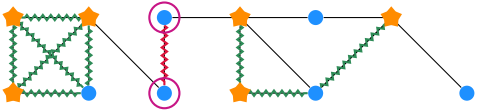

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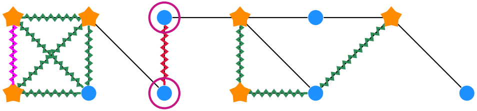
Note that the green edges denote $G[X]$.



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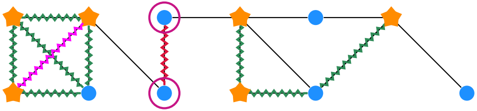

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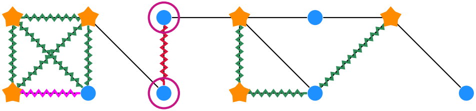

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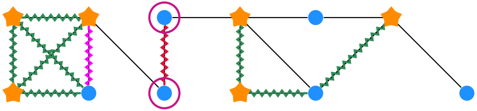

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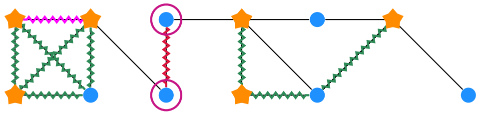

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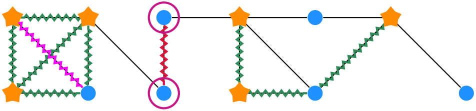

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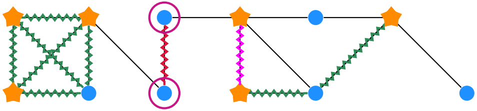

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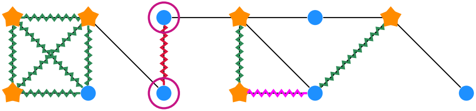
then there is a set $\widehat{e}$ in X such that $\widehat{e} \cap S = \emptyset$.

Note that the green edges denote $G[X]$.



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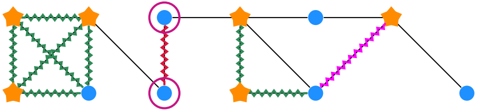

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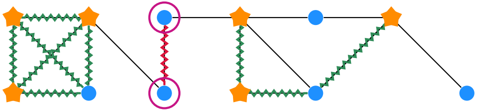

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Contradiction!

A k-representative family for $E(G)$ is in fact an $O(k^2)$ instance kernel for Vertex Cover!

REPRESENTATIVE SETS

*Why, What and **How.***

Notation

$$\mathrm{Det}(M) : [\![M]\!]$$

Let M be a $m \times n$ matrix, and let $I \subseteq [m]$, $J \subseteq [n]$.

$M[I, J]$: M restricted to rows indexed by I and columns indexed by J

$M[\star, J]$: M restricted to all rows and columns indexed by J

$M[I, \star]$: M restricted to rows indexed by I and all columns

STANDARD LAPLACE EXPANSION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a row and expand along the columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a row and expand along the columns.

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Fix a row and expand along the columns.

$$\begin{bmatrix} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & & a_{34} & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{35} & a_{36} \\ a_{51} & a_{52} & a_{53} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & & a_{35} & a_{36} \\ & & & & & \\ a_{51} & a_{52} & a_{53} & & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & & a_{36} \\ & & & & & \\ a_{51} & a_{52} & a_{53} & a_{54} & & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a row and expand along the columns.

$$\begin{bmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{51} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{34} & a_{35} & a_{36} \\ a_{51} & a_{52} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{35} & a_{36} \\ a_{51} & a_{52} & a_{53} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{65} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{36} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{66} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} \end{bmatrix}$$

GENERALIZED LAPLACE EXPANSION

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

$$\mathsf{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathsf{Det}(A[\bar{I}, \bar{J}]) \cdot \mathsf{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

$$\operatorname{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \operatorname{Det}(A[\bar{I}, \bar{J}]) \cdot \operatorname{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

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$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

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$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \quad \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix} (-1)^{(1+3+6)+(1+2+6)}$$

$\operatorname{Det}(A[\bar{I}, \bar{J}])$. $\operatorname{Det}(A[I, J])$.

$$\operatorname{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \operatorname{Det}(A[\bar{I}, \bar{J}]) \cdot \operatorname{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & & & & & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ & & & & & \\ a_{61} & & a_{63} & & & a_{66} \end{bmatrix} (-1)^{(1+3+6)+(1+2+6)}$$

$\text{Det}(A[\bar{I}, \bar{J}])$. $\text{Det}(A[I, J])$.

$$\text{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \text{Det}(A[\bar{I}, \bar{J}]) \cdot \text{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & a_{42} & & a_{44} & a_{45} & \\ & a_{52} & & a_{54} & a_{55} & \\ & a_{62} & & a_{64} & a_{65} & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ a_{31} & & a_{33} & & & a_{36} \\ & & & & & \\ & & & & & \\ & & & & & \end{bmatrix} (-1)^{(1+3+6)+(1+2+3)}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

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$$\begin{bmatrix} a_{32} & a_{34} & a_{35} \\ a_{52} & a_{54} & a_{55} \\ a_{62} & a_{64} & a_{65} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} & a_{16} \\ a_{21} & a_{23} & a_{26} \\ a_{41} & a_{43} & a_{46} \end{bmatrix} (-1)^{(1+3+6)+(1+2+4)}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} & & & & & \\ & & & & & \\ & a_{32} & & a_{34} & a_{35} & \\ & a_{42} & & a_{44} & a_{45} & \\ & & & & & \\ & a_{62} & & a_{64} & a_{65} & \end{bmatrix} \begin{bmatrix} a_{11} & & a_{13} & & & a_{16} \\ a_{21} & & a_{23} & & & a_{26} \\ & & & & & \\ & & & & & \\ a_{51} & & a_{53} & & & a_{56} \\ & & & & & \end{bmatrix} (-1)^{(1+3+6)+(1+2+5)}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{22} & a_{24} & a_{25} \\ a_{42} & a_{44} & a_{45} \\ a_{62} & a_{64} & a_{65} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} & a_{16} \\ a_{31} & a_{33} & a_{36} \\ a_{51} & a_{53} & a_{56} \end{bmatrix} (-1)^{(1+3+6)+(1+3+5)}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} a_{22} & a_{24} & a_{25} \\ a_{42} & a_{44} & a_{45} \\ a_{52} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} a_{11} & a_{13} & a_{16} \\ a_{31} & a_{33} & a_{36} \\ a_{61} & a_{63} & a_{66} \end{bmatrix} (-1)^{(1+3+6)+(1+3+6)}$$

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$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66}
\end{bmatrix}
$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$
\begin{bmatrix}
 & & & & & \\
 & a_{22} & & a_{24} & a_{25} & \\
 & a_{32} & & a_{34} & a_{35} & \\
 & & & & & \\
 & & & & & \\
 & a_{62} & & a_{64} & a_{65} &
\end{bmatrix}
\begin{bmatrix}
a_{11} & & a_{13} & & & a_{16} \\
 & & & & & \\
 & & & & & \\
a_{41} & & a_{43} & & & a_{46} \\
a_{51} & & a_{53} & & & a_{56} \\
 & & & & &
\end{bmatrix}
(-1)^{(1+3+6)+(1+4+5)}
$$

$\mathsf{Det}(A[\bar{I}, \bar{J}])$. $\mathsf{Det}(A[I, J])$.

$$
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$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

Fix a set of columns, $J \subseteq [6]$.

Iterate over all $I \subseteq [6]$ such that $|I| = |J|$.

$$\begin{bmatrix} a_{12} & a_{14} & a_{15} \\ a_{22} & a_{24} & a_{25} \\ a_{62} & a_{64} & a_{65} \end{bmatrix} \begin{bmatrix} a_{31} & a_{33} & a_{36} \\ a_{41} & a_{43} & a_{46} \\ a_{51} & a_{53} & a_{56} \end{bmatrix} (-1)^{(1+3+6)+(3+4+5)}$$

$\mathrm{Det}(A[\bar{I}, \bar{J}])$. $\mathrm{Det}(A[I, J])$.

$$\mathrm{Det}(A) = \sum_{I \subseteq [n], |I| = |J|} \mathrm{Det}(A[\bar{I}, \bar{J}]) \cdot \mathrm{Det}(A[I, J]) \cdot (-1)^{\sum I + \sum J}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

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Recall: A Linear (or Representable) Matroid

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

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$$A_{\mathcal{M}} = \overbrace{\begin{pmatrix} & & & & \\ & & & & \\ & & & & \end{pmatrix}}^{\text{Columns indexed by elements of } E}$$

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$$A_{\mathcal{M}} = \overbrace{\begin{pmatrix} \vdots & & \\ x_{e_1} & & \\ \vdots & & \end{pmatrix}}^{\text{Columns indexed by elements of } E}$$

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \overbrace{\begin{pmatrix} \vdots & \vdots & & \\ x_{e_1} & x_{e_2} & & \\ \vdots & \vdots & & \end{pmatrix}}^{\text{Columns indexed by elements of } E}$$

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$$A_M = \overbrace{\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & \\ x_{e_1} & x_{e_2} & \cdots & x_{e_i} & & \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix}}^{\text{Columns indexed by elements of } E}$$

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$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \left(\overbrace{\begin{array}{cccccc} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}}^{\text{Columns indexed by elements of } E} \right)$$

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \overbrace{\begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}}^{\text{Columns indexed by elements of } E}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

Columns corresponding to $S \in \mathcal{I}$

$$A_{\mathcal{M}} = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

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...are linearly independent.

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

Columns that are linearly independent...

$$A_M = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

$$\mathcal{M} = (E, \mathcal{I}), \text{ where } E = \{e_1, \ldots, e_n\} \text{ and } \mathcal{I} \subseteq 2^E$$

Columns that are linearly independent...

$$A_{\mathcal{M}} = \begin{pmatrix} x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \end{pmatrix}$$

...correspond to sets in $\mathcal{I}$.

$\mathcal{M} = (E, \mathcal{I})$, where $E = \{e_1, \ldots, e_n\}$ and $\mathcal{I} \subseteq 2^E$

$$A_M = \left.\overbrace{\begin{pmatrix} \vdots & \vdots & & \vdots & & \vdots & \vdots \\ x_{e_1} & x_{e_2} & \cdots & x_{e_i} & \cdots & x_{e_{n-1}} & x_{e_n} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \end{pmatrix}}^{\text{Columns indexed by elements of } E}\right\} \mathrm{rk}(M)$$

Given: A collection of p-sized independent sets[1]:

$$\mathcal{S} = \{S_1, \ldots, S_t\}.$$

[1] The rank of the underlying matroid is $(p + q)$.

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$$\mathcal{S} = \{S_1, \ldots, S_t\}.$$

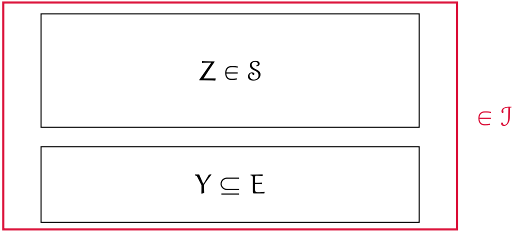
Want: A q-representative subfamily $\widehat{\mathcal{S}}$ of size $\leqslant \binom{p+q}{p}$.

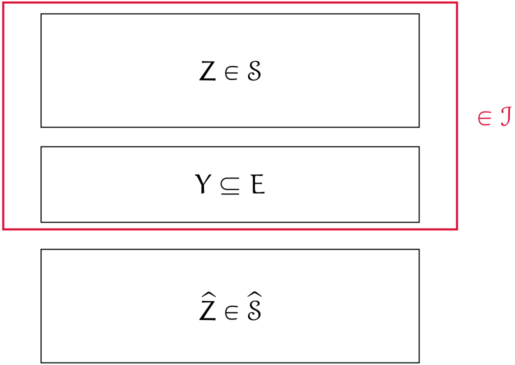
[1] The rank of the underlying matroid is $(p+q)$.

$$Z \in \mathcal{S}$$

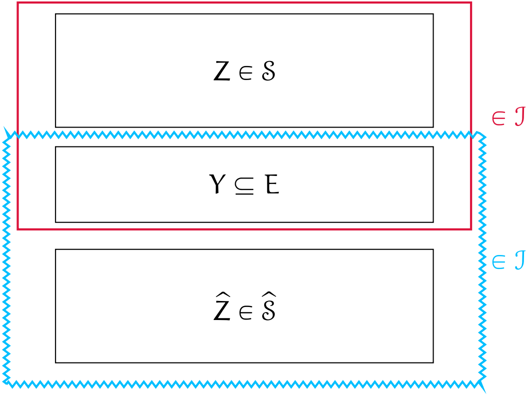
$$Z \in \mathcal{S}$$

$$Y \subseteq E$$


$Z \in \mathcal{S}$
$Y \subseteq E$
$\in \mathcal{I}$



$Z \in \mathcal{S}$
$Y \subseteq E$
$\in \mathcal{I}$
$\widehat{Z} \in \widehat{\mathcal{S}}$


$Z \in \mathcal{S}$
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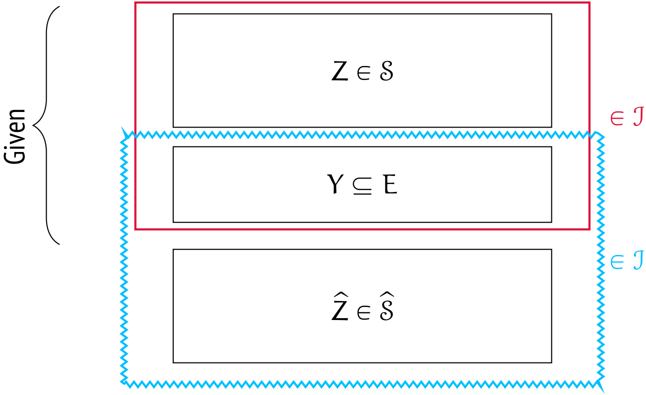
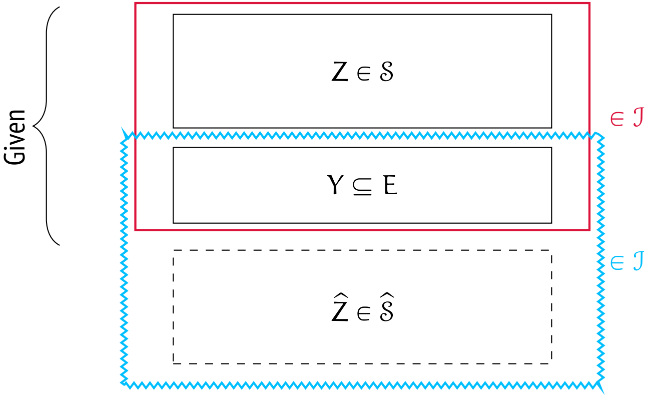
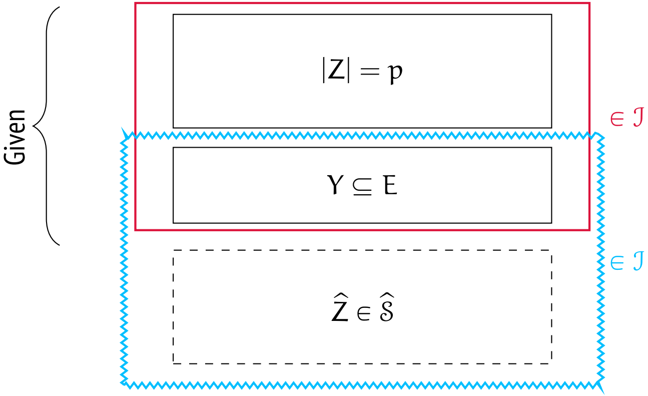
$\in \mathcal{I}$
$\in \mathcal{I}$

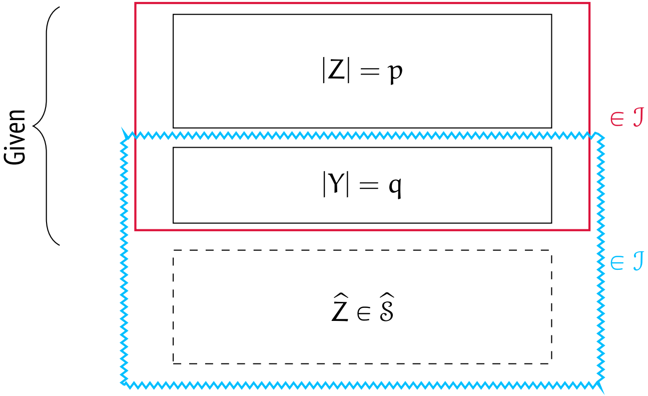


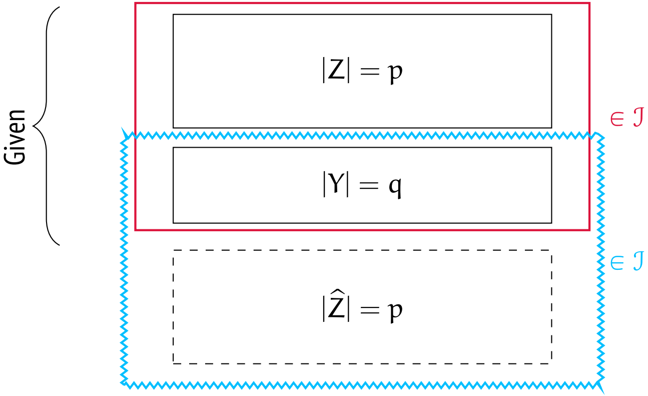
Given
$Z \in \mathcal{S}$
$Y \subseteq E$
$\hat{Z} \in \hat{\mathcal{S}}$
$\in \mathcal{I}$
$\in \mathcal{I}$


Given
$Z \in \mathcal{S}$
$Y \subseteq E$
$\in \mathcal{I}$
$\in \mathcal{I}$
$\widehat{Z} \in \widehat{\mathcal{S}}$



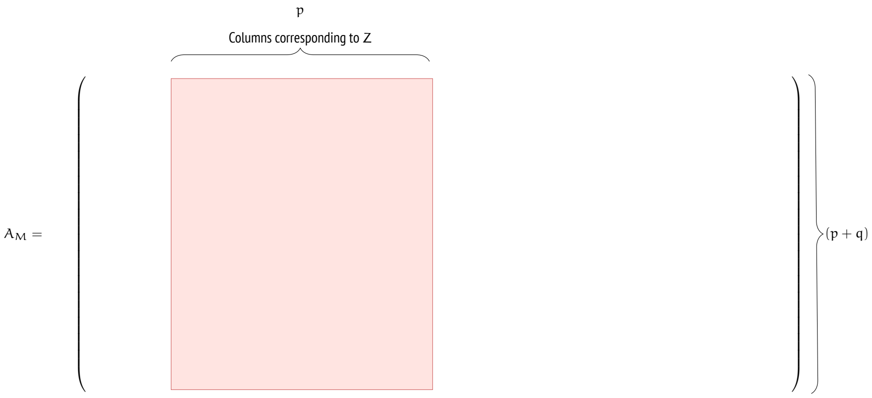
Given
|Z| = p
Y ⊆ E
∈ 𝒥
Ẑ ∈ Ŝ
∈ 𝒥


Given
|Z| = p
|Y| = q
∈ ℐ
∈ ℐ
Ẑ ∈ Ŝ

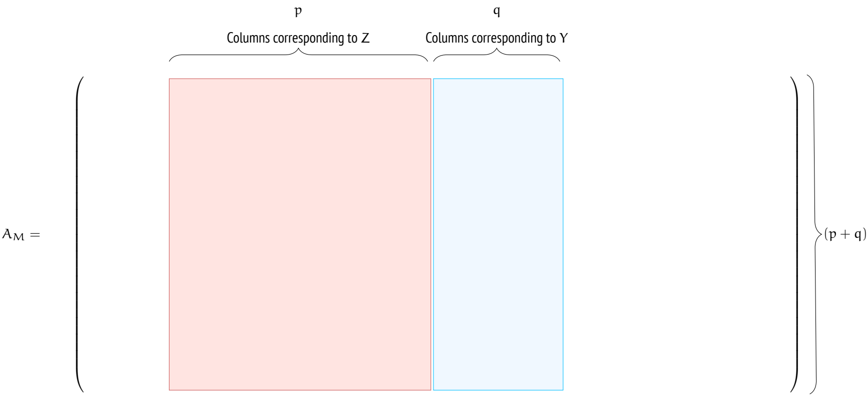


Given
$|Z| = p$
$|Y| = q$
$|\widehat{Z}| = p$
$\in \mathcal{J}$
$\in \mathcal{J}$

$$A_M = \left.\begin{pmatrix} \\ \\ \\ \end{pmatrix}\right\}(p+q)$$

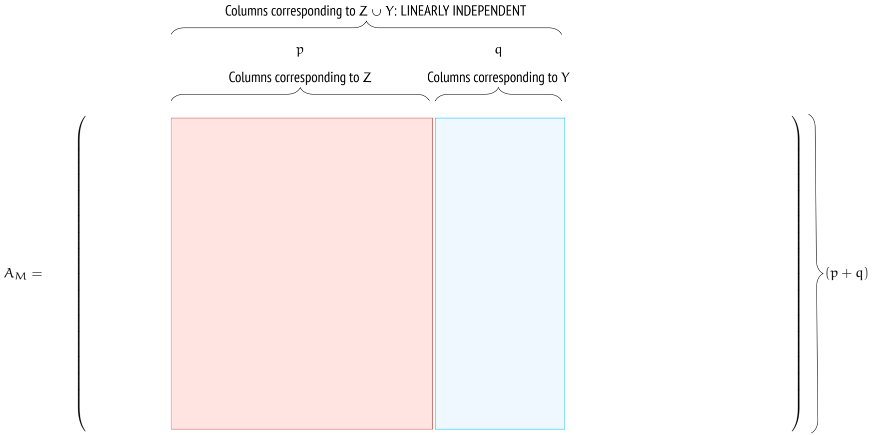

p
Columns corresponding to Z
A_M =
(p + q)

$A_M =$
p
Columns corresponding to Z
q
Columns corresponding to Y
$(p+q)$



$$\mathrm{Det}(A_M[\star, Z \cup Y])$$

$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y])$$



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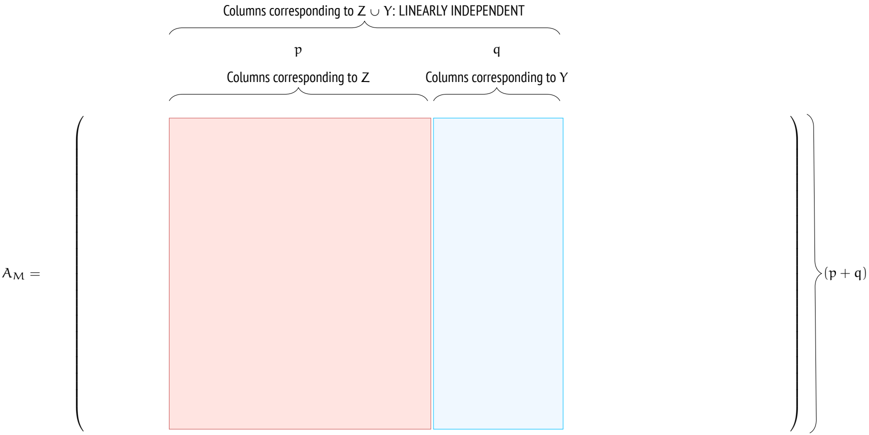
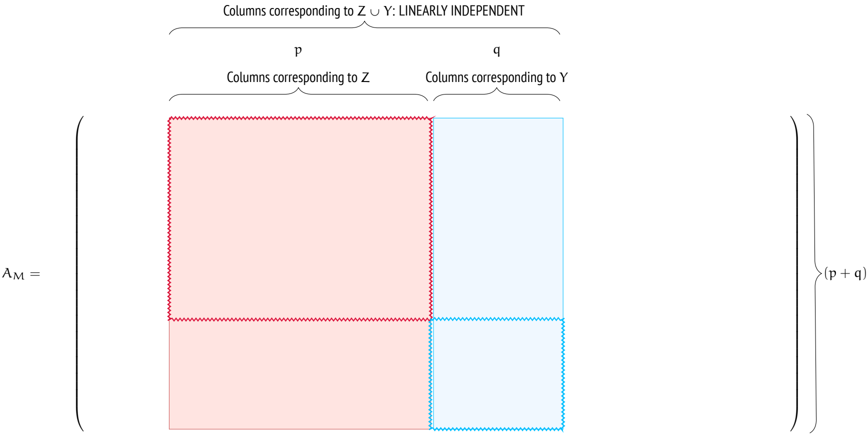


$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \mathrm{Det}(A[I, Z]) \cdot \mathrm{Det}(A[\bar{I}, Y]) \cdot \oslash$$



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$$v_Z := \left(\mathsf{Det}(A[I_0, Z]) \;, \ldots, \; \mathsf{Det}(A[I_j, Z]) \;, \ldots, \; \mathsf{Det}(A[I_r, Z]) \right)$$

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All subsets of size p of $(p + q)$.

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All subsets of size p of $(p+q)$.

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$$\vdots$$

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$$\vdots$$

$$v_{S_t} := \Big(\mathrm{Det}(A[I_0, S_t]) \;, \ldots, \; \mathrm{Det}(A[I_j, S_t]) \;, \ldots, \; \mathrm{Det}(A[I_r, S_t]) \Big)$$

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All subsets of size p of $(p+q)$.

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$$\vdots$$

$$v_{S_t} := \left(\mathsf{Det}(A[I_0, S_t]) \;, \ldots, \; \mathsf{Det}(A[I_j, S_t]) \;, \ldots, \; \mathsf{Det}(A[I_r, S_t]) \right)$$

$$\chi(\mathcal{S}) := \{v_{S_1}, \ldots\ldots\ldots, v_{S_i}, \ldots\ldots\ldots, v_{S_t}\}$$

$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \mathrm{Det}(A[I, Z]) \cdot \mathrm{Det}(A[\bar{I}, Y]) \cdot \oslash$$

$$v_Z := \left(\mathrm{Det}(A[I_0, Z]) \;, \ldots, \; \mathrm{Det}(A[I_j, Z]) \;, \ldots, \; \mathrm{Det}(A[I_r, Z]) \right)$$

All subsets of size p of $(p+q)$.

$$\mathcal{S} = \{S_1, \ldots\ldots\ldots, S_i, \ldots\ldots\ldots, S_t\}$$

$$v_{S_1} := \left(\mathrm{Det}(A[I_0, S_1]) \;, \ldots, \; \mathrm{Det}(A[I_j, S_1]) \;, \ldots, \; \mathrm{Det}(A[I_r, S_1]) \right)$$

$$\vdots$$

$$v_{S_i} := \left(\mathrm{Det}(A[I_0, S_i]) \;, \ldots, \; \mathrm{Det}(A[I_j, S_i]) \;, \ldots, \; \mathrm{Det}(A[I_r, S_i]) \right)$$

$$\vdots$$

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$\binom{p+q}{p}$-dimensional vectors.

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All subsets of size p of $(p+q)$.

$$\mathcal{S} = \{S_1, \ldots\ldots\ldots, S_i, \ldots\ldots\ldots, S_t\}$$

$$v_{T_1} := \Big(\operatorname{Det}(A[I_0, T_1]) \quad , \ldots , \quad \operatorname{Det}(A[I_j, T_1]) \quad , \ldots , \quad \operatorname{Det}(A[I_r, T_1]) \Big)$$

$$\vdots$$

$$v_{T_r} := \Big(\operatorname{Det}(A[I_0, T_r]) \quad , \ldots , \quad \operatorname{Det}(A[I_j, T_r]) \quad , \ldots , \quad \operatorname{Det}(A[I_r, T_r]) \Big)$$

A basis of size $\leqslant \binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \ldots\ldots\ldots, v_{S_i}, \ldots\ldots\ldots, v_{S_t}\}$$

$$0 \neq \operatorname{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I|=p} \operatorname{Det}(A[I, Z]) \cdot \operatorname{Det}(A[\bar{I}, Y]) \cdot \oslash$$

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$$v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r}$$

$$\boxed{\mathcal{S} = \{S_1, \ldots\ldots\ldots, S_i, \ldots\ldots\ldots, S_t\}}$$

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$$\vdots$$

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A basis of size $\leqslant \binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \ldots\ldots\ldots, v_{S_i}, \ldots\ldots\ldots, v_{S_t}\}$$

$$0 \neq \mathsf{Det}(A_M[\star, Z \cup Y]) = \sum_{I \subseteq [p+q], |I| = p} \sum_{i=1}^{r} \lambda_i \mathsf{Det}(A[I, T_i]) \cdot \mathsf{Det}(A[\bar{I}, Y]) \cdot \oslash$$

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$$\vdots$$

$$v_{T_r} := \begin{pmatrix} \mathsf{Det}(A[I_0, T_r]) & , \ldots, & \mathsf{Det}(A[I_j, T_r]) & , \ldots, & \mathsf{Det}(A[I_r, T_r]) \end{pmatrix}$$

A basis of size $\leqslant \binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \ldots\ldots\ldots, v_{S_i}, \ldots\ldots\ldots, v_{S_t}\}$$

$$0 \neq \operatorname{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \sum_{I \subseteq [p+q], |I|=p} \operatorname{Det}(A[I, T_i]) \cdot \operatorname{Det}(A[\bar{I}, Y]) \cdot \oslash$$

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$$\operatorname{Det}(A[I, Z]) = \lambda_1 \operatorname{Det}(A[I, T_1]) + \cdots + \lambda_r \operatorname{Det}(A[I, T_r])$$

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$$\vdots$$

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A basis of size $\leqslant \binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \dots\dots\dots, v_{S_i}, \dots\dots\dots, v_{S_t}\}$$

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$$\chi(\mathcal{S}) := \{v_{S_1}, \dots\dots\dots, v_{S_i}, \dots\dots\dots, v_{S_t}\}$$

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$$v_{T_1} := \Big(\ \mathsf{Det}(A[I_0, T_1]) \ \ , \ldots , \ \ \mathsf{Det}(A[I_j, T_1]) \ \ , \ldots , \ \ \mathsf{Det}(A[I_r, T_1]) \ \Big)$$

$$\vdots$$

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A basis of size $\leqslant \binom{p+q}{p}$ for

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$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \mathrm{Det}(A_M[\star, T_i \cup Y])$$

Note that at for at least one T_i, we have that:

$$\mathrm{Det}(A_M[\star, T_i \cup Y]) \neq 0$$

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For such a T_i, we know that:

1. $Y \cap T_i = \emptyset$ (easily checked: all terms that survive have this property),
2. $Y \cup T_i \in \mathcal{I}$ (since non-zero determinant $\longrightarrow$ linearly independent columns).

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For such a T_i, we know that:

1. $Y \cap T_i = \varnothing$ (easily checked: all terms that survive have this property),
2. $Y \cup T_i \in \mathcal{I}$ (since non-zero determinant $\longrightarrow$ linearly independent columns).

Thus, the sets corresponding to the basis vectors, $T_1, \ldots, T_r$, do form a q-representative family.

$$0 \neq \mathrm{Det}(A_M[\star, Z \cup Y]) = \sum_{i=1}^{r} \mathrm{Det}(A_M[\star, T_i \cup Y])$$

$$v_Z := \begin{pmatrix} \mathrm{Det}(A[I_0, Z]) & ,\dots, & \mathrm{Det}(A[I_j, Z]) & ,\dots, & \mathrm{Det}(A[I_r, Z]) \end{pmatrix}$$

$$v_Z = \lambda_1 v_{T_1} + \cdots + \lambda_r v_{T_r}$$

$$v_Z[I] = \lambda_1 v_{T_1}[I] + \cdots + \lambda_r v_{T_r}[I]$$

$$\mathrm{Det}(A[I, Z]) = \lambda_1 \mathrm{Det}(A[I, T_1]) + \cdots + \lambda_r \mathrm{Det}(A[I, T_r])$$

$$v_{T_1} := \begin{pmatrix} \mathrm{Det}(A[I_0, T_1]) & ,\dots, & \mathrm{Det}(A[I_j, T_1]) & ,\dots, & \mathrm{Det}(A[I_r, T_1]) \end{pmatrix}$$

$$\vdots$$

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A basis of size $\binom{p+q}{p}$ for

$$\chi(\mathcal{S}) := \{v_{S_1}, \dots\dots\dots, v_{S_i}, \dots\dots\dots, v_{S_t}\}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

Computing $T_1, \dots, T_r$.

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We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & & & & \\ v_{S_1} & & & & \\ \vdots & & & & \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & & & & & & \\ v_{S_1} & v_{S_2} & & & & & & \\ \vdots & \vdots & & & & & & \end{pmatrix}$$

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$$\begin{pmatrix} \vdots & \vdots & \vdots & & & & \\ v_{S_1} & v_{S_2} & \cdots & & & & \\ \vdots & \vdots & \vdots & & & & \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & & & & \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & & & & \\ \vdots & \vdots & \vdots & \vdots & & & & \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} | & | & | & | & | & | \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & v_{S_{t-1}} \\ | & | & | & | & | & | \end{pmatrix}$$

Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

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Computing $T_1, \ldots, T_r$.

We form a matrix with the vectors $\{v_{S_1}, \ldots, v_{S_i}, \ldots, v_{S_t}\}$ as the columns:

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{S_1} & v_{S_2} & \cdots & v_{S_i} & \cdots & v_{S_{t-1}} & v_{S_t} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

...and compute a column basis.

$$\begin{pmatrix} [\![A[I_0, S_1]]\!] & [\![A[I_0, S_2]]\!] & \dots & [\![A[I_0, S_i]]\!] & \dots & [\![A[I_0, S_t]]\!] \\ [\![A[I_1, S_1]]\!] & [\![A[I_1, S_2]]\!] & \dots & [\![A[I_1, S_i]]\!] & \dots & [\![A[I_1, S_t]]\!] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\![A[I_j, S_1]]\!] & [\![A[I_j, S_2]]\!] & \dots & [\![A[I_j, S_i]]\!] & \dots & [\![A[I_j, S_t]]\!] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\![A[I_r, S_1]]\!] & [\![A[I_r, S_2]]\!] & \dots & [\![A[I_r, S_i]]\!] & \dots & [\![A[I_r, S_t]]\!] \end{pmatrix}$$

$$\begin{pmatrix} \overbrace{\begin{matrix} [\![A[I_0, S_1]]\!] & [\![A[I_0, S_2]]\!] & \dots & [\![A[I_0, S_i]]\!] & \dots & [\![A[I_0, S_t]]\!] \end{matrix}}^{\text{t columns}} \\ \begin{matrix} [\![A[I_1, S_1]]\!] & [\![A[I_1, S_2]]\!] & \dots & [\![A[I_1, S_i]]\!] & \dots & [\![A[I_1, S_t]]\!] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\![A[I_j, S_1]]\!] & [\![A[I_j, S_2]]\!] & \dots & [\![A[I_j, S_i]]\!] & \dots & [\![A[I_j, S_t]]\!] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [\![A[I_r, S_1]]\!] & [\![A[I_r, S_2]]\!] & \dots & [\![A[I_r, S_i]]\!] & \dots & [\![A[I_r, S_t]]\!] \end{matrix} \end{pmatrix}$$

$$
\overbrace{\begin{pmatrix}
\llbracket A[I_0, S_1]\rrbracket & \llbracket A[I_0, S_2]\rrbracket & \dots & \llbracket A[I_0, S_i]\rrbracket & \dots & \llbracket A[I_0, S_t]\rrbracket \\
\llbracket A[I_1, S_1]\rrbracket & \llbracket A[I_1, S_2]\rrbracket & \dots & \llbracket A[I_1, S_i]\rrbracket & \dots & \llbracket A[I_1, S_t]\rrbracket \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\llbracket A[I_j, S_1]\rrbracket & \llbracket A[I_j, S_2]\rrbracket & \dots & \llbracket A[I_j, S_i]\rrbracket & \dots & \llbracket A[I_j, S_t]\rrbracket \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\llbracket A[I_r, S_1]\rrbracket & \llbracket A[I_r, S_2]\rrbracket & \dots & \llbracket A[I_r, S_i]\rrbracket & \dots & \llbracket A[I_r, S_t]\rrbracket
\end{pmatrix}}^{t \text{ columns}}
\left.\vphantom{\begin{pmatrix}a\\a\\a\\a\\a\\a\end{pmatrix}}\right\} \binom{p+q}{q} \text{ rows}
$$

$t \cdot \binom{p+q}{q}$ Determinant Computations.

Let $\mathcal{M}$ be a linear matroid of rank $p + q = k$, $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a p-family of independent sets. Then there exists a q-representative of size at most $\binom{p+q}{q}$.

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Moreover, given a representation of $\mathcal{M}$ over a field $\mathbb{F}$, we can find such a representative family in $O\left(\binom{p+q}{q}tp^{\omega} + t\binom{p+q}{q}^{\omega-1}\right)$ operations over $\mathbb{F}$.

REPRESENTATIVE SETS

And that will be all!