Parameterized Algorithms

Lecture 2: Introduction cont...

May 15, 2020

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Recall....

Vertex Cover

- **Input :** Graph G on n vertices and integer k
- **Parameter :** k
- **Output :** $S \subseteq V(G)$ such that $G - S$ is edgeless and $|S| \leq k$.

Iterative Compression

Input

A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a vertex cover of size at most k?

Question

Vertex Cover

Input A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a vertex cover of size at most k? **Question**

A vertex cover of size $(k+1)$.

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Let us "guess" how a vertex cover of size at most k interacts with this one.

VERTEX COVER

Input A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k? **Question**

A vertex cover of size (k+1).

Let us "guess" how a vertex cover of size at most k interacts with this one.

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Let us "guess" how a vertex cover of size at most k interacts with this one.

VERTEX COVER

Input A graph G with a vertex cover of size k+1.

Question Is there a vertex cover of size at most k?

A vertex cover of size (k+1).

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Can there be an edge between two red vertices?

VERTEX COVER

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Vertex Cover

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

Question

A vertex cover of size (k+1).

Now we need to make up for the work that the red vertices were doing.

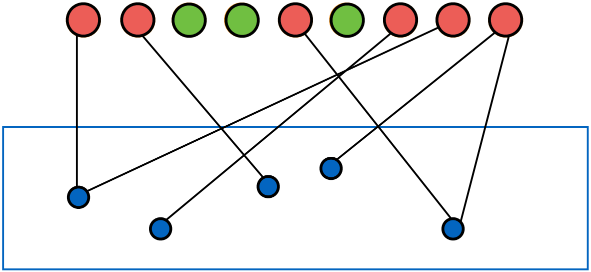
VERTEX COVER

Input

A graph G with a vertex cover of size k+1.

Is there a vertex cover of size at most k?

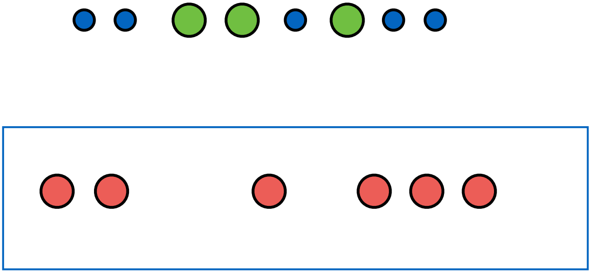
Question



Vertex Cover

Input A graph G with a vertex cover of size k+1.

Question Is there a vertex cover of size at most k?



Vertex Cover

Input

A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a vertex cover of size at most k?

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Is there a vertex cover of size at most k? **Question**

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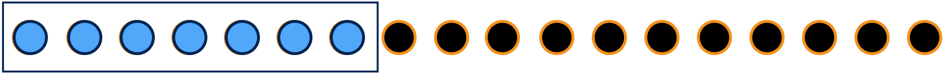

The first $k+2$ vertices in G.

Vertex Cover

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Is there a vertex cover of size at most k?

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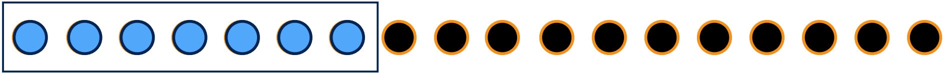

The first $k+2$ vertices in G. It has an easy vertex cover of size $k+1$.

Vertex Cover

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Is there a vertex cover of size at most k?

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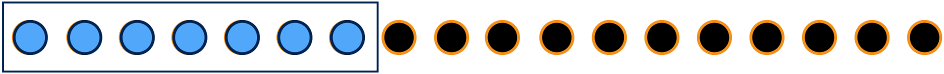

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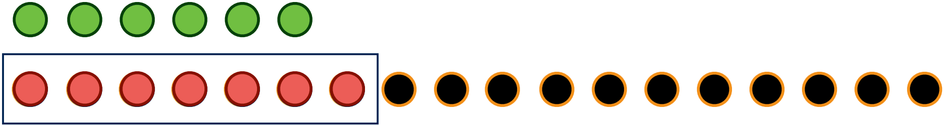

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Vertex Cover

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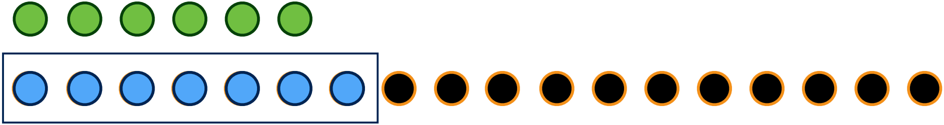

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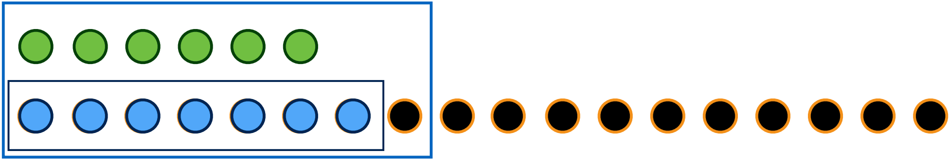

The first k+2 vertices in G. It has an easy vertex cover of size k+1.

Vertex Cover

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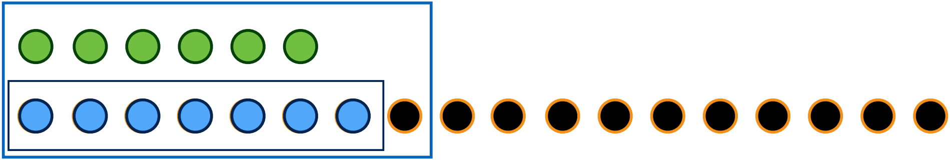

The first k+2 vertices in G. It has an easy vertex cover of size k+1.

Compress again… rinse, repeat.

Vertex Cover

Input

A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges?

Question

We have a $O^*(2^k)$ algorithm for Vertex Cover.

Vertex Cover

Recall....

Feedback Vertex Set

- **Input :** Graph G on n vertices and integer k
- **Parameter :** k
- **Output :** $S \subseteq V(G)$ such that $G - S$ is acyclic and $|S| \leq k$.

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

A Feedback Vertex Set of size (k+1).

Let us "guess" how a FVS of size at most k interacts with this one.

Feedback Vertex Set

Input A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic? **Question**

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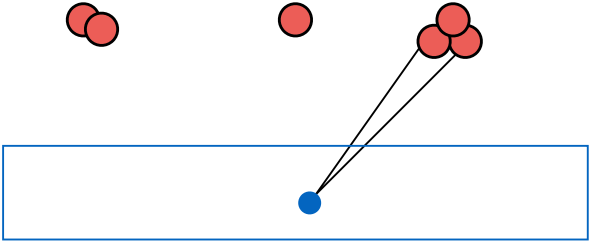

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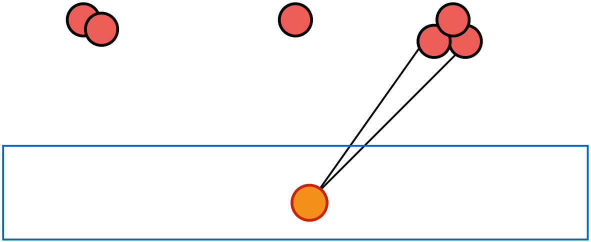

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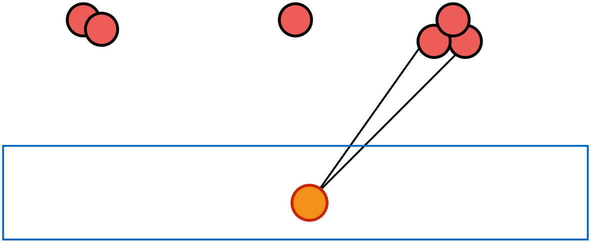

A vertex with two neighbors in the same component is forced.

FEEDBACK VERTEX SET

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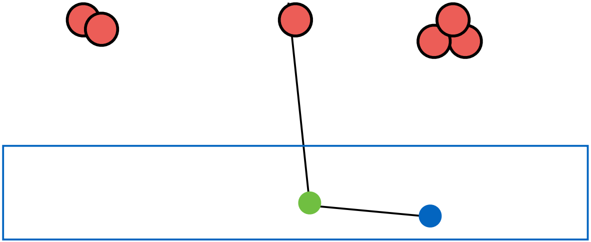

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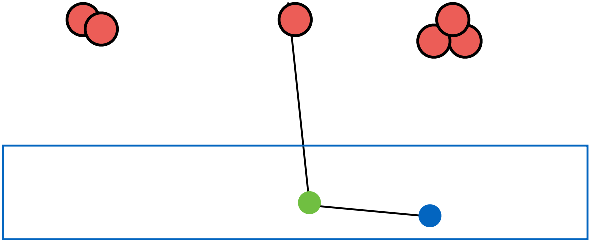

A leaf that has exactly one neighbor above can be preprocessed.

FEEDBACK VERTEX SET

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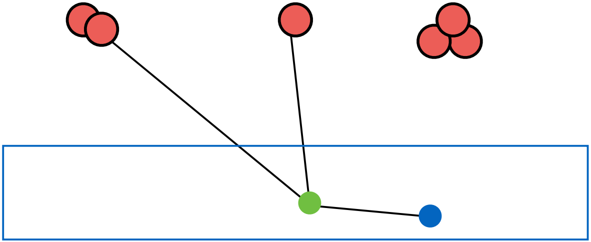
Question



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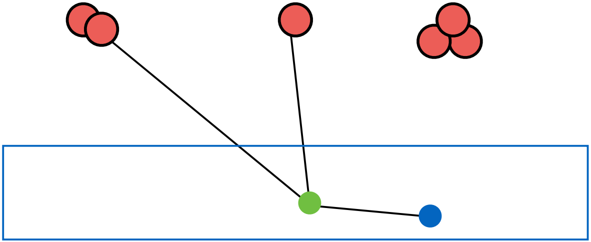

...a leaf with at least two neighbors in different components.

Feedback Vertex Set

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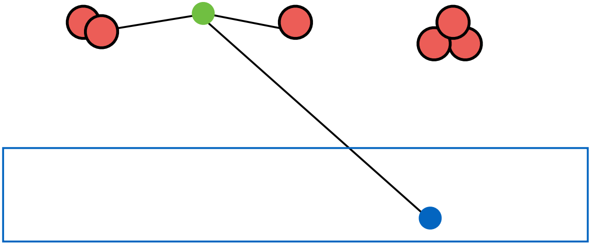
Is there a subset of at most k vertices whose removal makes the graph acyclic? **Question**



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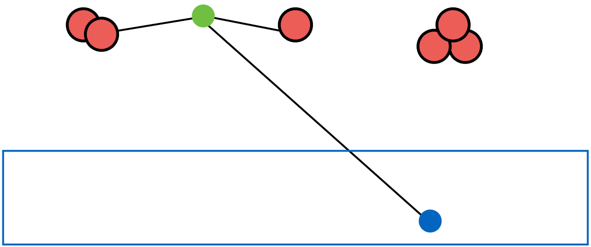


The leaf merges two components when we don't include it.

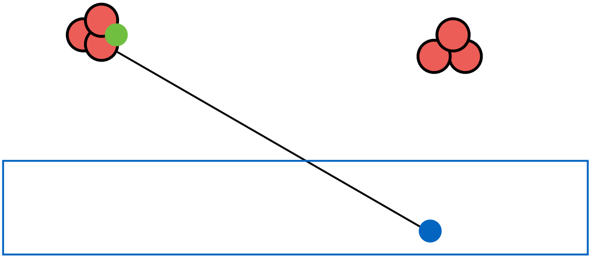
FEEDBACK VERTEX SET

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question



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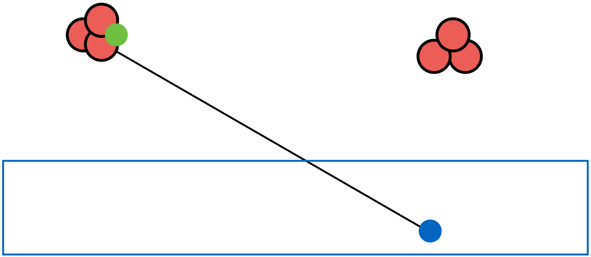

The number of components "on top" decreases.

Feedback Vertex Set

Input A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic? **Question**

Start with a leaf.

Two neighbors in one component: forced.
At most one neighbor above: preprocess.
At least two neighbors, all in different components above: branch.

Feedback Vertex Set

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

FEEDBACK VERTEX SET

Input A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic? **Question**

Let t denote the number of components among the red vertices. Let w = (k+t).

Feedback Vertex Set

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

Let t denote the number of components among the red vertices. Let w = (k+t).

Include v… k drops by 1
Exclude v… t drops by at least 1

Feedback Vertex Set

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

Let t denote the number of components among the red vertices. Let w = (k+t).

Include v… k drops by 1
Exclude v… t drops by at least 1

Either way, w drops by at least one.

FEEDBACK VERTEX SET

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

Either way, w drops by at least one.

Feedback Vertex Set

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

Either way, w drops by at least one.

Running time: $2^w = 2^{(k+t)}$

Feedback Vertex Set

Input

A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic?

Question

Either way, w drops by at least one.

Running time: $2^w = 2^{(k+t)} \leq 2^{k+k} \leq 4^k$

Feedback Vertex Set

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Either way, w drops by at least one.

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Overall Running Time…

Feedback Vertex Set

Input A graph on n vertices, m edges and an integer k.

Is there a subset of at most k vertices whose removal makes the graph acyclic? **Question**

Either way, w drops by at least one.

Running time: $2^w = 2^{(k+t)} \leq 2^{k+k} \leq 4^k$

Overall Running Time...

$$\sum_{i=1}^{k} \binom{k}{i} 4^k = 5^k$$

Feedback Vertex Set

Input

A tournament T on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic?

Question

This implies an $O(5^k)$ algorithm for FVS.

Feedback Vertex Set

Bipartite Deletion

Also Known as ODD CYCLE TRANSVERSAL

- **Input :** Graph G on n vertices and integer k
- **Parameter :** k
- **Output :** $S \subseteq V(G)$ such that $G - S$ is Bipartite and $|S| \leq k$.

Bipartite Graphs:

- Are 2-colorable
- Have no odd cycle

BIPARTITE DELETION

Solution based on iterative compression:

- **Step 1:**

  Solve the **annotated problem** for bipartite graphs:

  Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

BIPARTITE DELETION

Solution based on iterative compression:

- **Step 1:**
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- **Step 2:**
  Solve the **compression problem** for general graphs:

  Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

Bipartite Deletion

Solution based on iterative compression:

- **Step 1:**
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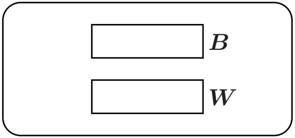
  Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

- **Step 3:**
  Apply the magic of iterative compression...

Step 1: The annotated problem

Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.



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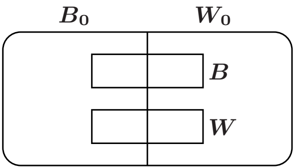
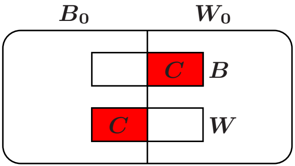


Find an arbitrary 2-coloring (B_0, W_0) of G.

Step 1: The annotated problem

Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.



Find an arbitrary 2-coloring (B_0, W_0) of G.
$C := (B_0 \cap W) \cup (W_0 \cap B)$ should change color, while
$R := (B_0 \cap B) \cup (W_0 \cap W)$ should remain the same color.

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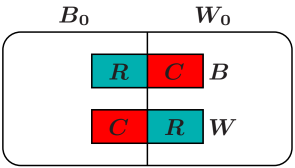

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$C := (B_0 \cap W) \cup (W_0 \cap B)$ should change color, while
$R := (B_0 \cap B) \cup (W_0 \cap W)$ should remain the same color.

Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Step 1: The annotated problem

Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

Proof:

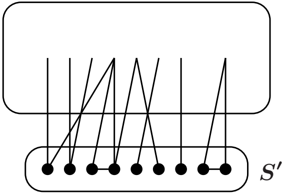
$\Rightarrow$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

Step 2: The compression problem

Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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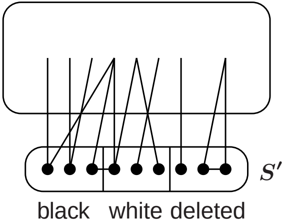

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.
Trivial check: no edge between two black or two white vertices.

Step 2: The compression problem

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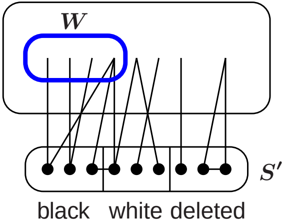

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

Neighbors of the black vertices in S' should be white and the neighbors of the white vertices in S' should be black.

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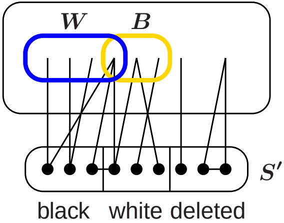

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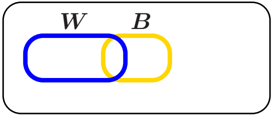

The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

How do we get a solution of size $k + 1$?

Step 3: Iterative compression

How do we get a solution of size $k+1$?

We get it for free!

Step 3: Iterative compression

How do we get a solution of size $k+1$?

We get it for free!

Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k+1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression($G_i, S_{i-1} \cup \{v_i\}$)
5. If the answer is "NO" $\Rightarrow$ return "NO"
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time
$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Feedback Vertex Set in Tournaments

- **Input :** A tournament D and integer k.
- **Parameter :** k
- **Output :** $S \subseteq V(D)$ such that $D - S$ is an acyclic directed graph.

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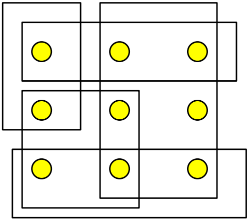
Tournament:

- A Complete Directed Graph.
- Exercise: There is a (directed) cycle if and only if there is a cycle of length 3.
- Exercise: Using iterative compression, show that FVST is FPT parameterized by the solution size k.

Dynamic Programming

Set Cover



- **Input :** A universe U of n elements and a family $\mathcal{F} = \{F_1, F_2, \ldots F_m\}$ of m subsets of U.
- **Parameter :** n (Size of the Universe)
- **Output :** A subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of minimum size that "covers U"

$$\bigcup_{F \in \mathcal{F}'} F = U$$

Theorem

SET COVER *is FPT parameterized by the size of the universe*

Running time: $2^n \cdot \mathsf{poly}(n, m)$

Set Cover: Dynamic Programming

- Fix an ordering of the family $\mathcal{F}$: $F_1, F_2, \ldots, F_m$
- Dynamic Programming Table,

$$\text{for every } X \subseteq U \text{ and } j \in \{0, 1, \ldots, m\}$$

$$T[X, j] = \text{ size of a min subset of } \{F_1, F_2, \ldots, F_j\} \text{ that covers } X$$

- Table Size: $2^{|U|} \times m$
- **Base Case :** $T[X, 0] = 0$ if $X = \emptyset$, else it is ∞.
- **Recursize Step :**

$$T[X, j] = \min \left\{ T[X, j-1],\ \ 1 + T[X \setminus F_j, j-1] \right\}$$

  - Either X can be covered using within $\{F_1, F_2, \ldots, F_{j-1}\}$
  - Or we need F_j + best solution of $X \setminus F_j$
- $T[U, m]$ is the minimum set cover size
- Maintain a candidate solution along with each $T[X, j]$.

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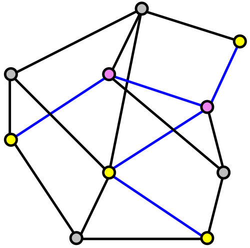
- $T[U, m]$ is the minimum set cover size
- Maintain a candidate solution along with each $T[X, j]$.

Exercise: Prove that $T[X, j]$ indeed contains a minimum set cover of X from $\{F_1, F_2, \ldots, F_j\}$

Steiner Tree



- **Input :** Graph G on n vertices, $S \subseteq V(G)$ of k vertices called Terminals.
- **Parameter :** k (Number of Terminals)
- **Output :** Minimum connected subgraph H of G that contains all of S.

Observation : H must be a Tree

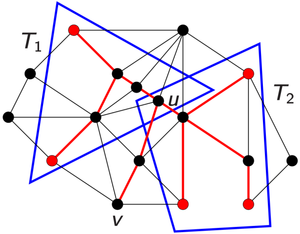
Theorem

Steiner Tree *can be solved in time* $3^k \cdot \mathsf{poly}(n)$.

Notation:

- $d_G(u, v) =$ length of shortest path between u and v in G.
- Assume every terminal $s \in S$ has degree 1

Steiner Tree: Dynamic Programming



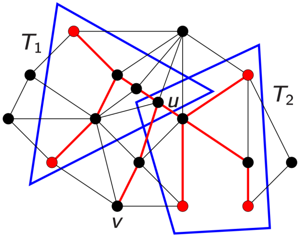
- DP Table: For $X \subseteq S$ and $v \in V(G)$

  $T[X, v] =$ minimum cost of a sub-tree containing $X \cup v$.

- Table Size: $2^S \cdot n$
- Base Case I: $T[\emptyset, v] = 1$ for every $v \in V(G)$
- Base Case II: $T[\{s\}, v] = d_G(s, v)$ for every $s \in S$
- Recursive Case: for $X \subseteq V(G)$, $|X| \geq 2$

$$T[X, v] = \min_{u \in V(G),\ \emptyset \neq Y \subsetneq X} d_G(u, v) + T[Y, u] + T[X \setminus Y, u]$$

Steiner Tree: Dynamic Programming



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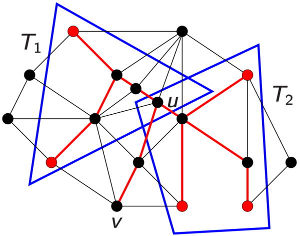
Correctness: (LHS $\leq$ RHS)

- For any $Y \subseteq X$ and $u \in V(G)$, the RHS is the cost of a sub-tree connecting $X \cup v$.
- RHS = min-cost subtree for $Y \cup u$ + min-cost subtree for $(X \setminus Y) \cup u$ + shortest path between u and v

Steiner Tree: Dynamic Programming



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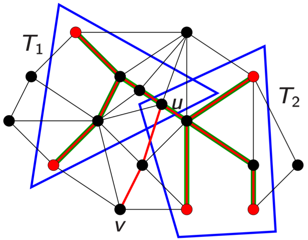
Correctness: (LHS $\geq$ RHS)

- Consider a minimum subtree H of G connecting $X \cup v$.
- root H at v, and u is the closest descendant with multiple children $\{u_1, u_2, \ldots, u_\ell\}$

Note: u exists because $|X| \geq 2$ and all terminals have degree 1.
Further $d_H(u, v) = d_G(u, v)$, by choice of H

Steiner Tree: Dynamic Programming



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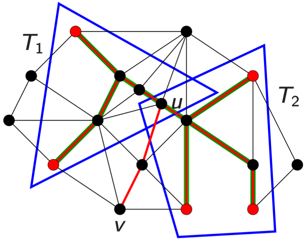
Correctness: (LHS $\geq$ RHS)

- Let Y = all terminal from X in sub-tree of u_1.
- Split H into 3 parts
  - The sub-path between u and v
  - The sub-tree of H rooted at u_1 + edge (u, u_1)
  - The sub-tree of H excluding the above

Steiner Tree: Dynamic Programming



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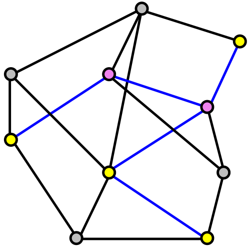
Running Time:

- Computing $T[X, v]$ requires $2^{|X|} \cdot \mathsf{poly}(n)$ time.
- Computing the entire table requires time:

$$\sum_{v \in V(G), X \subseteq S} 2^{|X|} \cdot \mathsf{poly}(n)$$

- This is $3^{|S|} \cdot \mathsf{poly}(n)$

Steiner Tree



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Theorem

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Exercise: Steiner Tree with weights (Positive Integers)

Thank You.

Iterative Compression slides,
courtesy Neeldhara Misra and Daniel Marx.