Parameterized Algorithms

Lecture 4: Kernelization

May 29, 2020

Max-Planck Institute for Informatics, Germany.

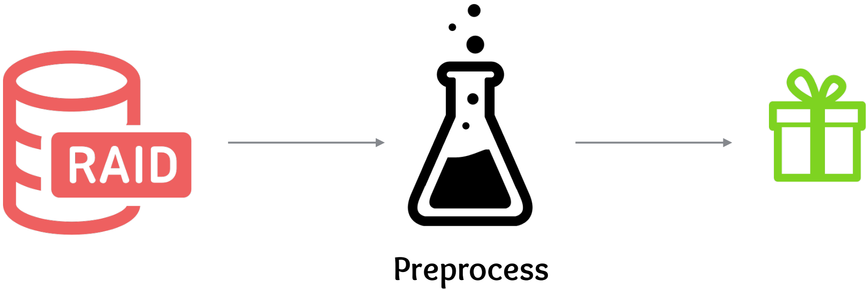
Kernelization

$(x, k) \longrightarrow (x', k')$

FAST

SMALL

SANE

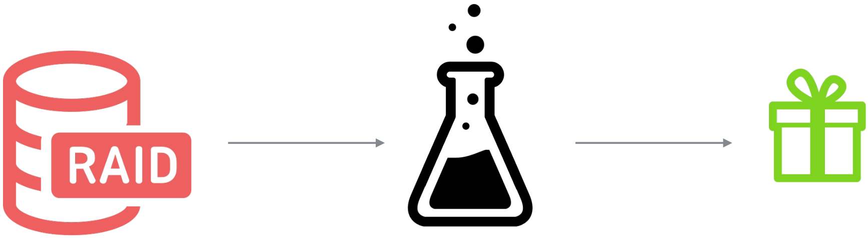


$$(x, k) \longrightarrow (x', k')$$

$|x|^{O(1)}$ time

SMALL

SANE



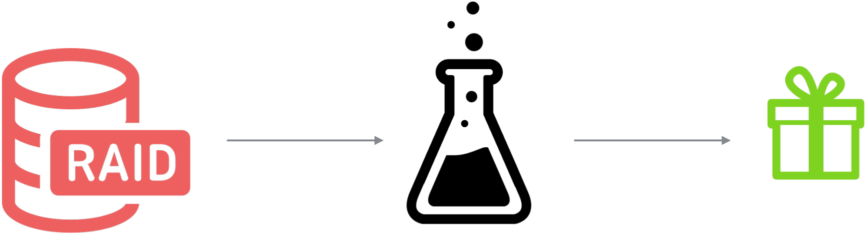
Preprocess

$(x, k) \longrightarrow (x', k')$

$|x|^{O(1)}$ **time**

$|x'| = f(k)$ **and** $k \leqslant k'$

SANE



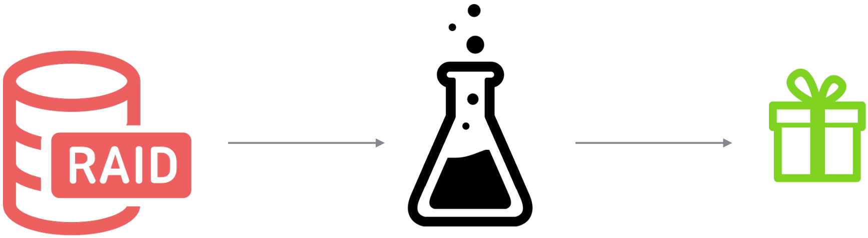
Preprocess

$$(x, k) \longrightarrow (x', k')$$

$|x|^{O(1)}$ **time**

$|x'| = f(k)$ **and** $k \leqslant k'$

$$(x, k) \equiv (x', k')$$



Preprocess

Max Sat

Input A CNF Formula over n variables with m clauses.

Max Sat

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? Question

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Is there an assignment satisfying at least k clauses? Question

What if k is at most m/2?

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Question

What if k is at most m/2?

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Say YES.

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What if k is at most m/2?

Say YES.

k > m/2?

Max Sat

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Is there an assignment satisfying at least k clauses? Question

What if k is at most m/2?

Say YES.

k > m/2?

The number of clauses is bounded by 2k.

Max Sat

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? Question

VARIABLES

Max Sat

Input — A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? — Question

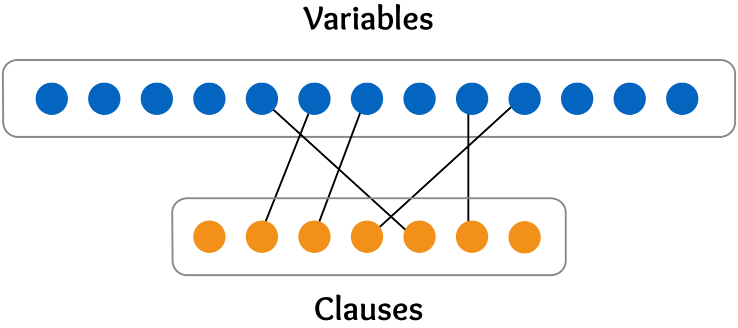
GOAL

We have at most k variables left.

Max Sat

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? **Question**



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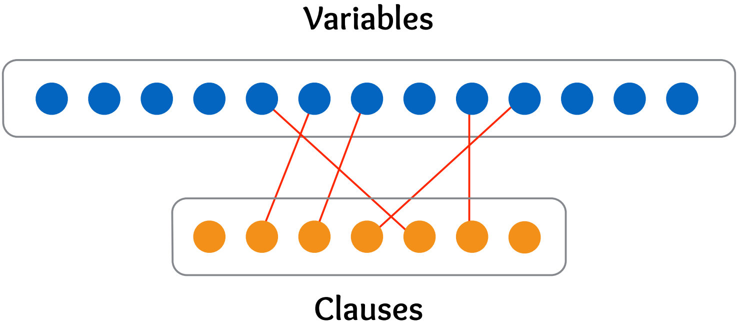


Max Sat

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? Question

If we have at most k variables - nothing to do.

If we have at least k variables and we have a matching from Variables — Clauses then we can say YES.

Max Sat

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? Question

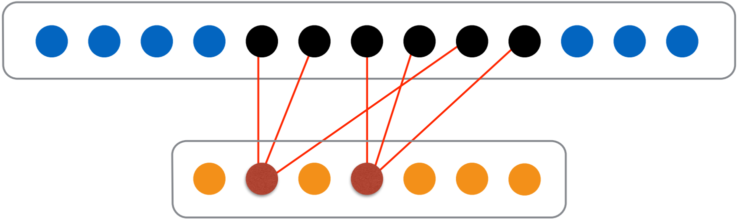
Else, we have at least k variables,
but no matching from Variables — Clauses.

Max Sat

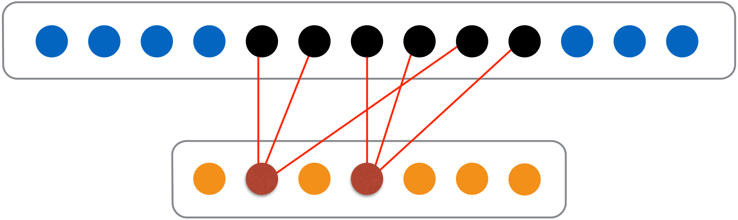
Hall's Theorem

If, in a bipartite graph with parts A and B,
there is no matching from A to B,
then there is a subset X of A
such that

$$|N(X)| < |X|$$

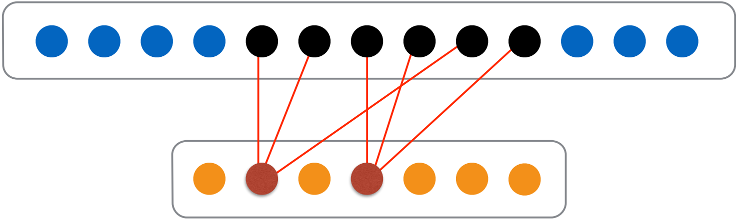


Hall's Theorem



Hall's Theorem

Such an "obstructing set" can be computed in polynomial time.



Hall's Theorem

[inclusion-minimal]

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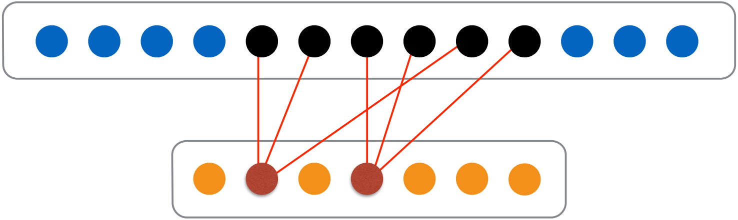

HALL'S THEOREM

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Is there an assignment satisfying at least k clauses? Question

[inclusion-minimal]

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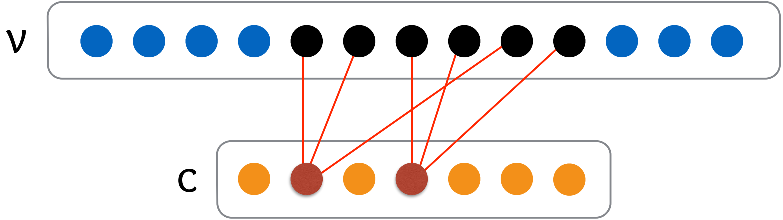

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[inclusion-minimal]

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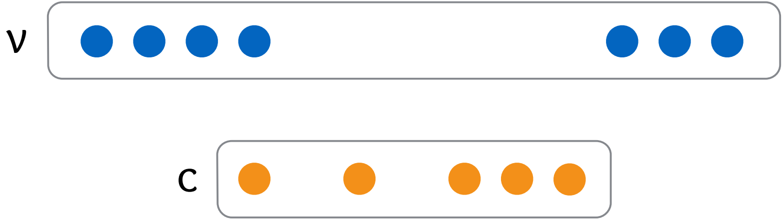

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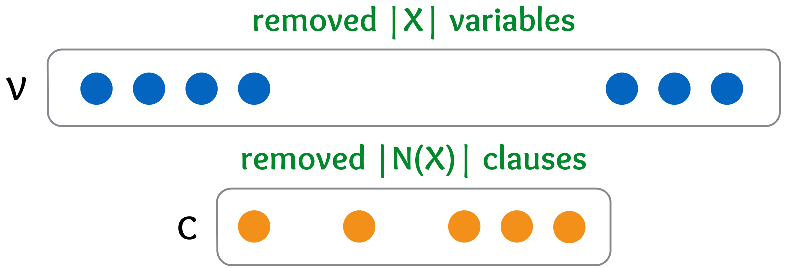

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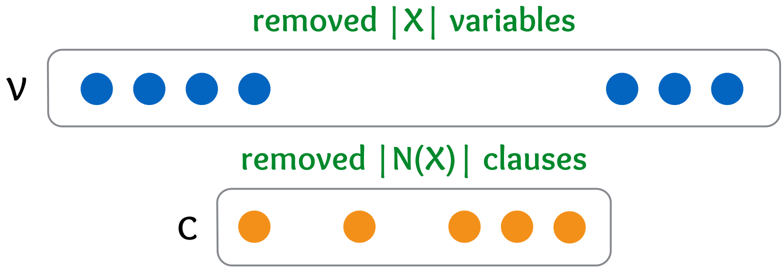
Is there an assignment satisfying at least k clauses? **Question**

Ask now if $k - |N(X)|$ clauses can be satisfied.



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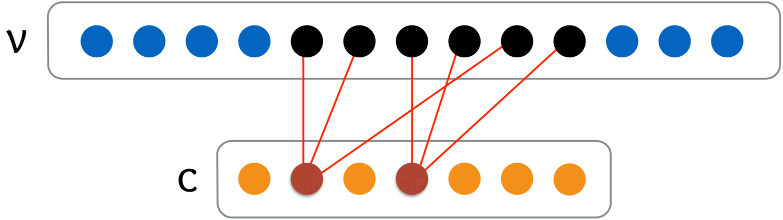

Max Sat

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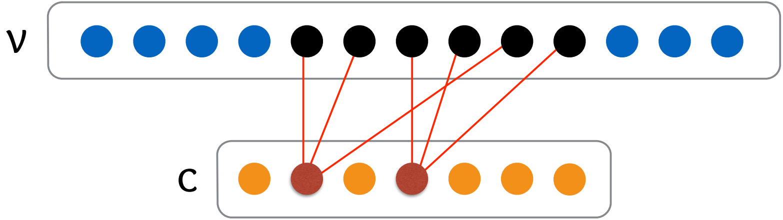
Is there an assignment satisfying at least k clauses? **Question**

We removed an inclusion-minimal violating set.



Max Sat

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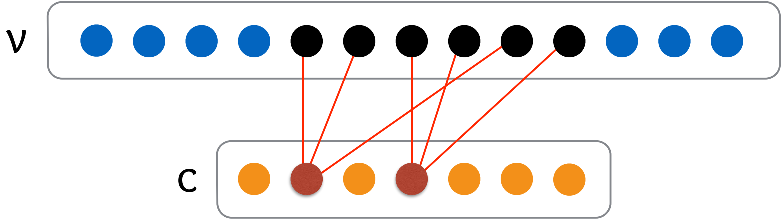
Is there an assignment satisfying at least k clauses? Question

Get rid of one vertex…



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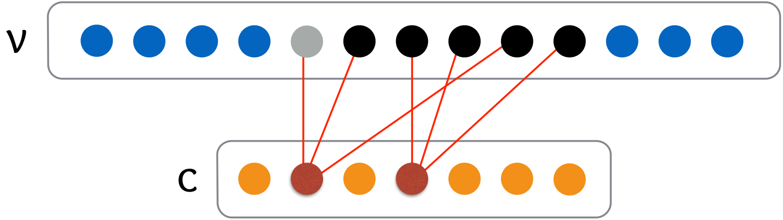


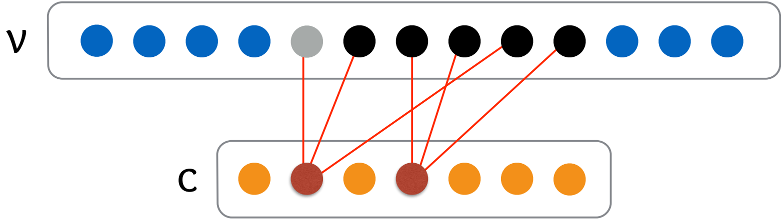
MAX SAT

Input — A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? — Question

Get rid of one vertex...

The rest of it can be matched!



MAX SAT

Input A CNF Formula over n variables with m clauses.

Is there an assignment satisfying at least k clauses? **Question**

This implies a kernel with at most k variables and 2k clauses.

MAX SAT

Vertex Cover

Input A graph G = (V,E) with n vertices, m edges, and k.

VERTEX COVER

Input

A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges?

Question

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What if a vertex has more than k neighbors?

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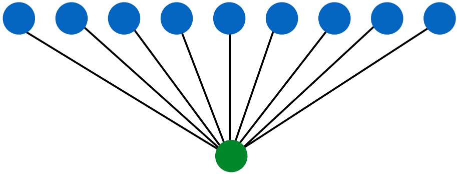

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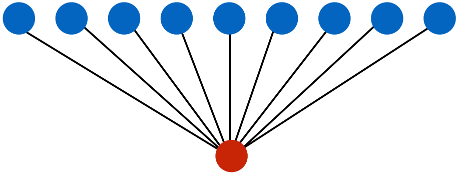

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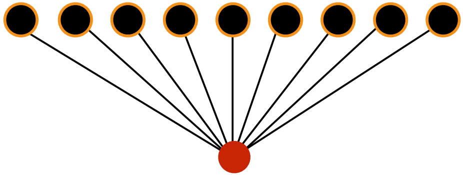

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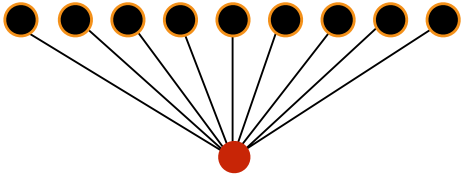
Question

What if a vertex has more than k neighbors?

We cannot afford to leave v out of any vertex cover of size at most k.



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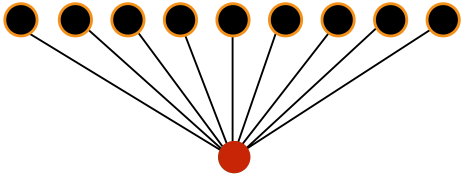

Vertex Cover

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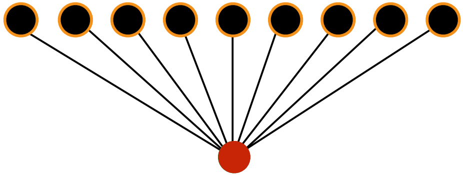
A graph G = (V,E) with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges?

Question

If a vertex has more than k neighbors,

delete it from the graph and reduce the budget by one.



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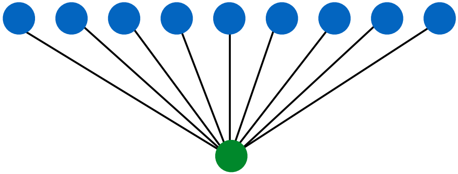
When we have nothing more to do...



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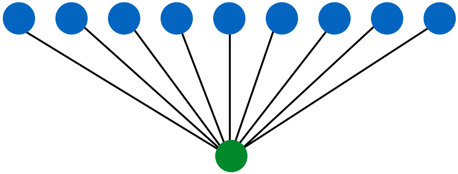
Is there a subset of vertices S of size at most k that intersects all the edges? **Question**

When we have nothing more to do...

every vertex has degree at most k.



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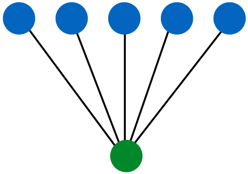
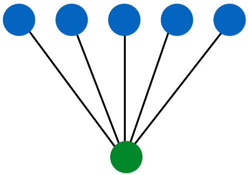


VERTEX COVER

Input A graph G = (V,E) with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges? **Question**

If the graph has more than k^2 edges,



VERTEX COVER

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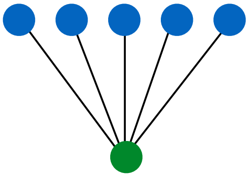
Is there a subset of vertices S of size at most k that intersects all the edges? Question

If the graph has more than k^2 edges,

reject the instance.



Vertex Cover

Input A graph G = (V,E) with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges? **Question**

Otherwise:

Vertex Cover

Input

A graph $G = (V,E)$ with n vertices, m edges, and k.

Is there a subset of vertices S of size at most k that intersects all the edges?

Question

Otherwise:

the number of edges is at most k^2.

Vertex Cover

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Vertices?

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Is there a subset of vertices S of size at most k that intersects all the edges? **Question**

Otherwise:

the number of edges is at most k^2.

Vertices?

k^2 edges can be involved in at most $2k^2$ vertices.
Throw away isolated vertices.

VERTEX COVER

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A graph $G = (V,E)$ with n vertices, m edges, and k.

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Question

This implies a kernel with at most $2k^2$ vertices and k^2 edges.

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[BUSS KERNELIZATION]

Vertex Cover

Feedback Arc Set on Tournaments

Input

A tournament on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic?

Question

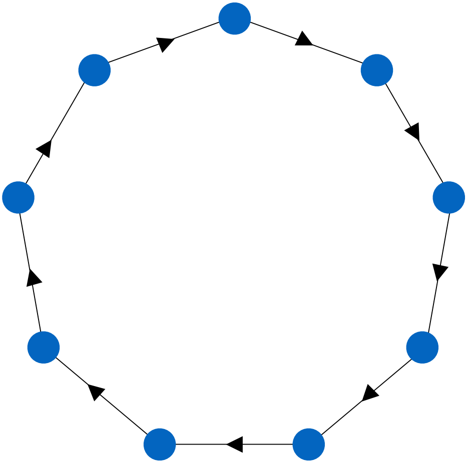
Feedback Arc Set on Tournaments

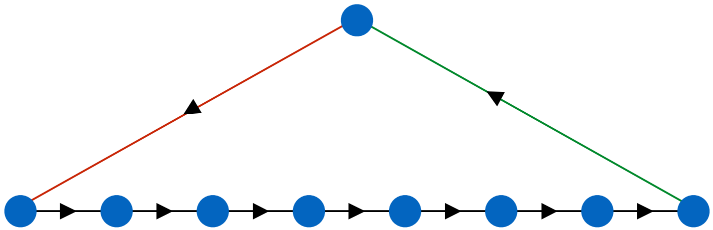
Input A tournament on n vertices and an integer k.

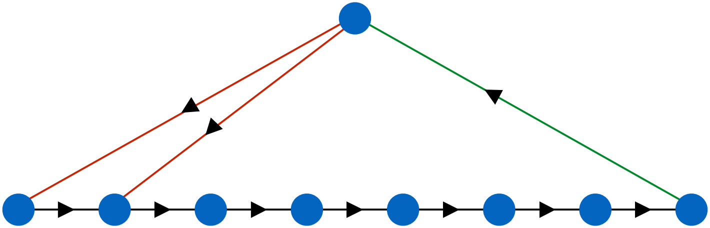
Is there a subset of k arcs that can be reversed to make the tournament acyclic? **Question**

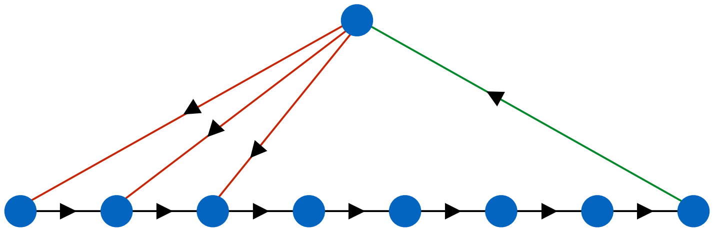
A tournament has a cycle if, and only if, it has a triangle.

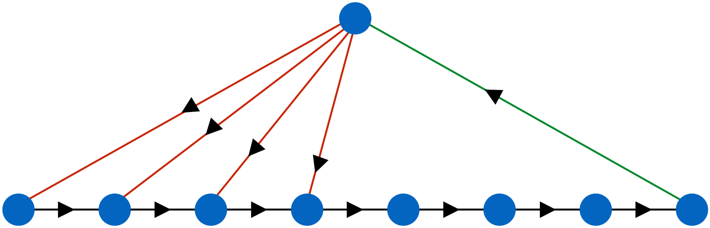
Feedback Arc Set on Tournaments

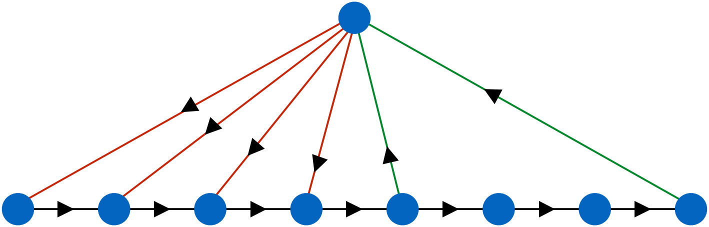












Input

A tournament on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic?

Question

A tournament has a cycle if, and only if, it has a triangle.

Delete a vertex if it does not belong to any triangle.

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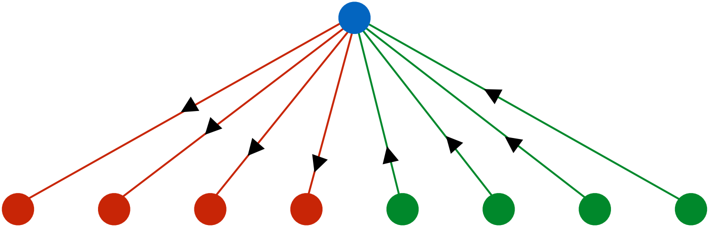

FEEDBACK ARC SET ON TOURNAMENTS

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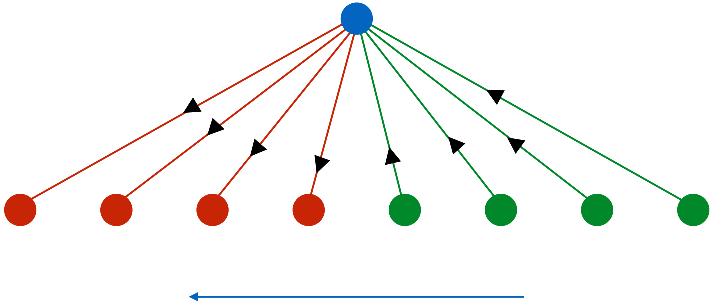

FEEDBACK ARC SET ON TOURNAMENTS

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Is there a subset of k arcs that can be reversed to make the tournament acyclic? **Question**

What about an edge that participates in more than k triangles?

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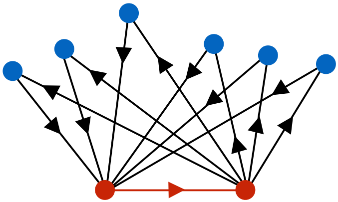

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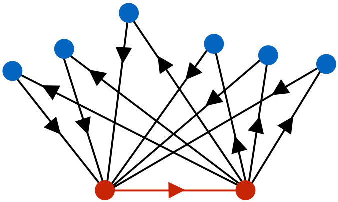

Reverse it and decrease the budget by one.

Feedback Arc Set on Tournaments

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At most k arcs in any solution.

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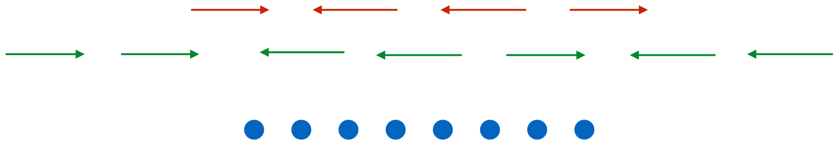

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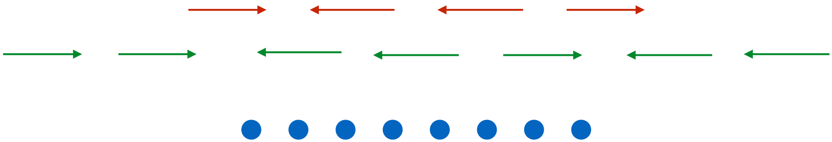
At most k arcs in any solution.



Every vertex is in a triangle, and every arc “sees” at most k triangles.

Feedback Arc Set on Tournaments

Input A tournament T on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic? **Question**

Feedback Arc Set on Tournaments

Input

A tournament T on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic?

Question

After reducing the graph, if there are more than $k^2 + 2k$ vertices,

reject the instance.

Feedback Arc Set on Tournaments

Input A tournament T on n vertices and an integer k.

Is there a subset of k arcs that can be reversed to make the tournament acyclic? Question

This implies a kernel with at most $k^2 + 2k$ vertices.

FEEDBACK ARC SET ON TOURNAMENTS

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F? Question

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What if there are more than k sets with one element in common?

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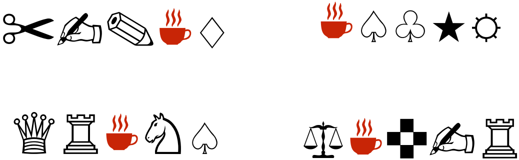

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What if there are more than k sets with one element in common, and every set is mutually disjoint otherwise?

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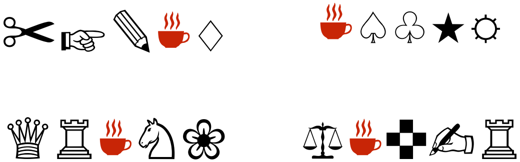

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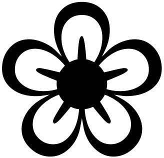

Sunflower Lemma

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F? **Question**

{1,2,53,54,55}
{1,2,68,69,67}
{1,2,15,12,11}
{1,2,23,24,29}
{1,2,72,71,70}

Sunflower Lemma

d-Hitting Set

Input

A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

Question

If a d-uniform family F has at least $d!(k-1)^d$ sets, then it has a subcollection of at least k sets that:

have a common mutual intersection (a core)

are pairwise disjoint otherwise (petals).

Sunflower Lemma

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F? Question

Sunflowers can also be computed in polynomial time.

Sunflower Lemma [By the way...]

d-Hitting Set

Input

A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

Question

As long as there is a sunflower involving at least k+1 petals, remove it and replace it with its core.

Intuition

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F? Question

{1,2,53,54,55}
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Intuition

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

Question

{1,2}

Intuition

d-Hitting Set

Input A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

Question

The rule destroys d-uniformity.

What if the core is empty?

Sunflower Lemma

d-Hitting Set

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A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

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The rule destroys d-uniformity.

Fix this by iterating over d.

What if the core is empty?

Sunflower Lemma

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Question

The rule destroys d-uniformity.

Fix this by iterating over d.

What if the core is empty?

This means we have more than k disjoint sets, so we can reject the instance.

Sunflower Lemma

d-Hitting Set

Input

A family F of d-sized sets over an universe U, and k.

Does U have a subset S of size at most k that intersects every set in F?

Question

This implies a kernel with at most $d!k^d$ sets, and $d(d!)k^d$ elements.

d-Hitting Set

Feedback Vertex Set

Problem Definition

FEEDBACK VERTEX SET **Parameter:** k
Input: An undirected graph G and a positive integer k.
Question: Does there exists a subset X of size at most k such that $G - X$ is acyclic?

X is called feedback-vertex set (fvs) of G.

Problem Definition

FEEDBACK VERTEX SET **Parameter:** k

Input: An undirected graph G and a positive integer k.

Question: Does there exists a subset X of size at most k such that $G - X$ is acyclic?

X is called feedback-vertex set (fvs) of G.

Goal is to obtain a polynomial kernel for FEEDBACK VERTEX SET.

What reduction rules we already know?

Reduction.FVS

If there is a loop at a vertex v*, delete* v *from the graph and decrease* k *by one.*

What reduction rules we already know?

Multiplicity of a multiple edge does not influence the set of feasible solutions to the instance (G, k).

Reduction.FVS

If there is an edge of multiplicity larger than 2, reduce its multiplicity to 2.

What reduction rules we already know?

Any vertex of degree at most 1 does not participate in any cycle in G, so it can be deleted.

Reduction.FVS

If there is a vertex v *of degree at most* 1*, delete* v*.*

What reduction rules we already know?

Concerning vertices of degree 2, observe that, instead of including into the solution any such vertex, we may as well include one of its neighbors.

Reduction.FVS

If there is a vertex v *of degree* 2*, delete* v *and connect its two neighbors by a new edge.*

What do we achieve after all these?

After exhaustively applying these four reduction rules, the resulting graph G

(P1) contains no loops,

(P2) has only single and double edges, and

(P3) has minimum vertex degree at least 3.

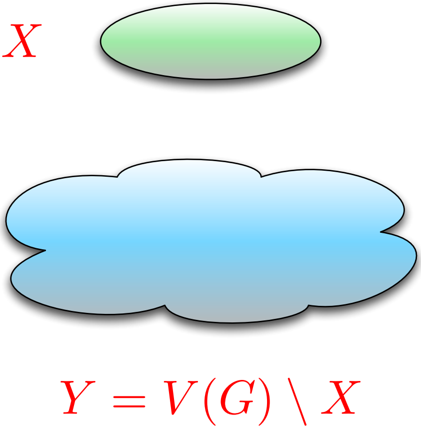
Stopping rule.

A rule that stops the algorithm if we already exceeded our budget.

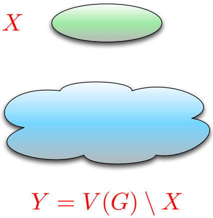
Reduction.FVS

If $k < 0$*, terminate the algorithm and conclude that* (G, k) *is a no-instance.*

A picture :)

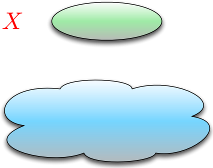


Maximum degree is **d**.



Maximum degree is **d**.

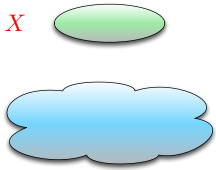
X



$Y = V(G) \setminus X$

$$\sum_{v \in V(G)} \text{degree}(v) = 2|E(G)|$$

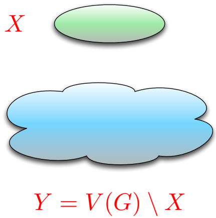
Maximum degree is **d**.



$Y = V(G) \setminus X$

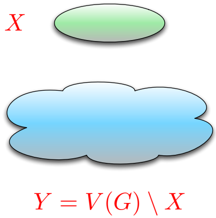
$$3|V(G)| \leqslant \sum_{v \in V(G)} \text{degree}(v) = 2|E(G)|$$

Maximum degree is **d**.



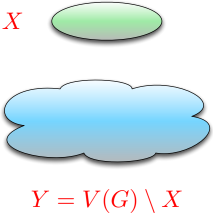
$$1.5|V(G)| \leqslant |E(G)|$$

Maximum degree is **d**.



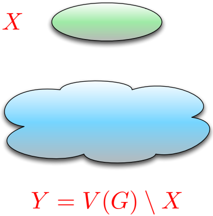
$$|E(G)| \leqslant d|X| + (|V(G)| - |X| - 1)$$

Maximum degree is **d**.



$$1.5|V(G)| \leqslant |E(G)| \leqslant d|X| + (|V(G)| - |X|)$$

Maximum degree is **d**.



$$1.5|V(G)| \leqslant |E(G)| \leqslant d|X| + (|V(G)| - |X|)$$

$$\implies |V(G)| \leqslant 2(d-1)|X| \leqslant 2(d-1)k.$$

Summarizing:

Lemma

If a graph G *has minimum degree at least* 3*, maximum degree at most* d*, and feedback vertex set of size at most* k*, then it has less than* $2(d-1)k$ *vertices and less than* $2(d-1)dk$ *edges.*

Summarizing: (possible to prove)

Lemma

If a graph G has minimum degree at least 3, maximum degree at most d, and feedback vertex set of size at most k, then it has less than $(d+1)k$ vertices and less than $2dk$ edges.

A new rule

Reduction.FVS

If $|V(G)| \geqslant (d+1)k$ *or* $|E(G)| \geqslant 2dk$*, where* d *is the maximum degree of* G*, then terminate the algorithm and return that* (G, k) *is a no-instance.*

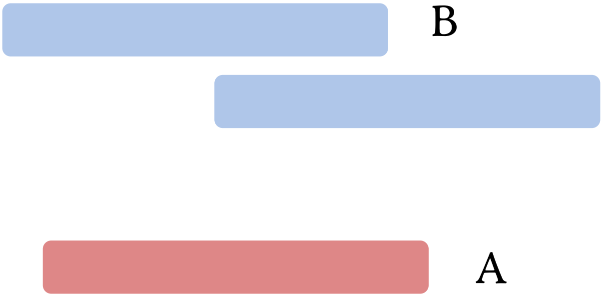
So what do we need to get the polynomial kernel?

So what do we need to get the polynomial kernel?

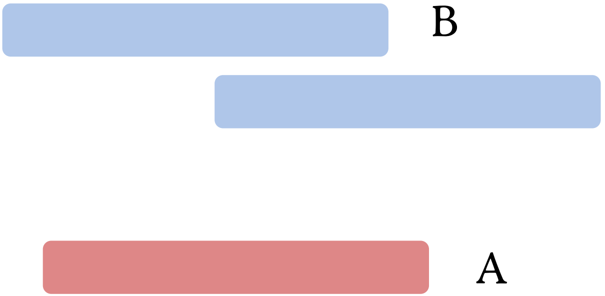
Bound the maximum degree of the graph by a polynomial in k.

Part 2: Recap

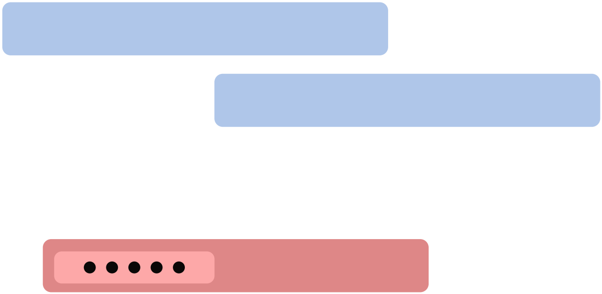
A Tale of 2 Matchings



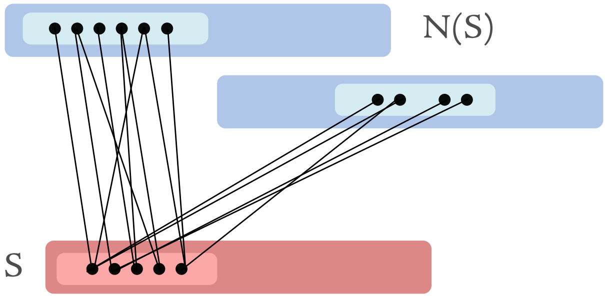
Consider a bipartite graph one of whose parts (say B) is at least twics as big as the other (call this A).



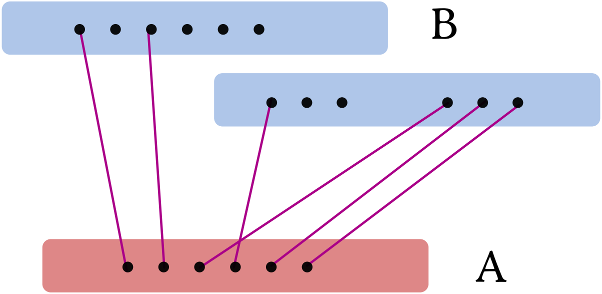
Assume that there are no isolated vertices in B.



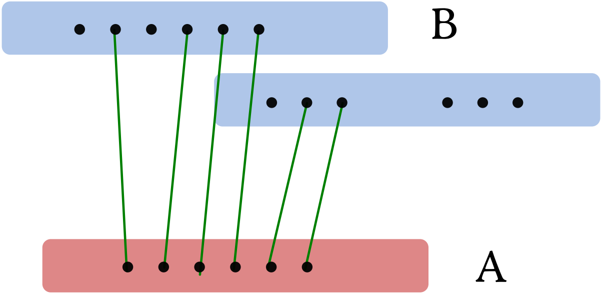
Suppose, further, that for every subset S in A,



Suppose, further, that for every subset S in A, $\mathrm{N}(S)$ is at least twice as large as $|S|$.



Then there exist two matchings saturating A,



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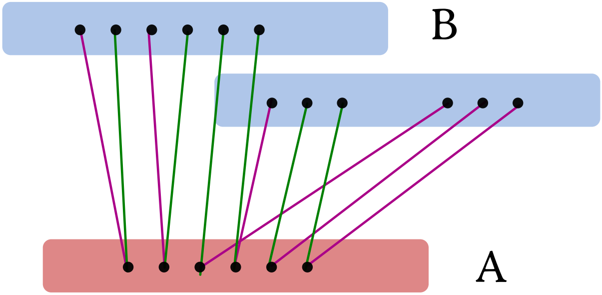

Then there exist two matchings saturating A,
and disjoint in B.

Claim:

If $|\mathrm{B}| \geqslant 2|A|$, then there exists a subset X of A such that:

there exists 2 matchings saturating the subset X that are vertex-disjoint in B.

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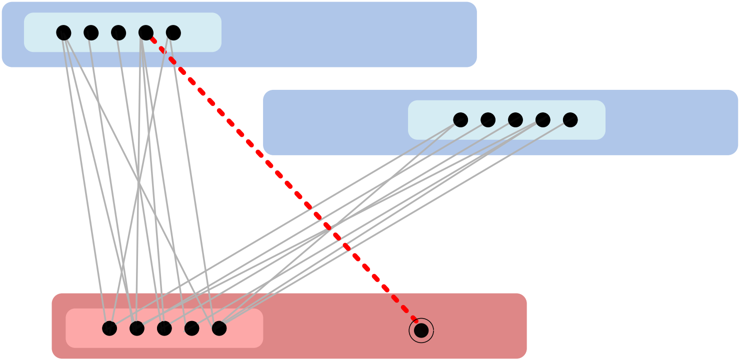
Claim:

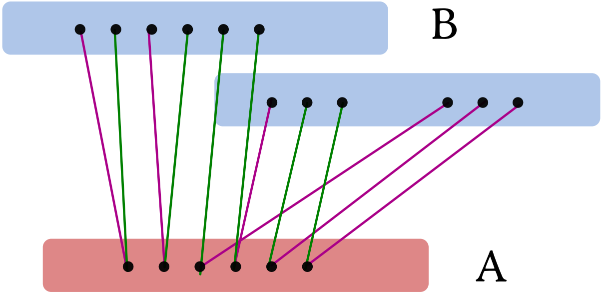
If $|B| \geqslant 2|A|$, then there exists a subset X of A such that:

$$\text{there exists 2 matchings saturating the subset } X \text{ that are vertex-disjoint in } B,$$

provided B does not have any isolated vertices.

Crucially: it turns out that the endpoints of the matchings in B (the larger set) do not have neighbors outside X.




B
A

q-Expansion Lemma

Let $q \geqslant 1$ be a positive integer and G be a bipartite graph with vertex bipartition (A, B) such that

(i) $|B| \geqslant q|A|$, and
(ii) there are no isolated vertices in B.

Then there exist nonempty vertex sets $X \subseteq A$ and $Y \subseteq B$ such that

- there is a q-expansion of X into Y, and
- no vertex in Y has a neighbor outside X, that is, $N(Y) \subseteq X$.

Furthermore, the sets X and Y can be found in time polynomial in the size of G.

q-Expansion Lemma

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Furthermore, the sets X and Y can be found in time polynomial in the size of G.

We will use this lemma with $q = 2$.

Part 3

2-Expansions and FVS

- For VERTEX COVER – if a vertex has degree $k + 1$ then we must have it in the solution.

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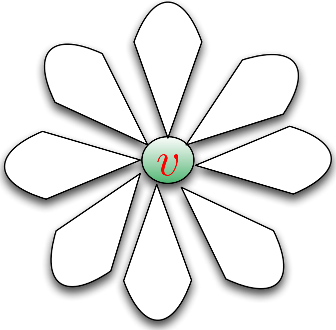
What would be the analogous rule for FEEDBACK VERTEX SET.

- For Vertex Cover – if a vertex has degree $k+1$ then we must have it in the solution.

What would be the analogous rule for Feedback vertex Set.

For Vertex Cover – wanted to hit edges and for Feedback vertex Set – want to hit cycles..

Flower



$k+1-$ vertex disjoint cycles passing through it

Flower Rule.

Reduction.FVS

If there is a $k+1$*-flower passing through a vertex* v *then* $(G \setminus \{v\}, k-1)$.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

A subset whose removal makes the graph acyclic.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

A polynomial function of k.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

Find an approximate feedback vertex set T.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

If T does not contain v, we are done.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

Else: $v \in \mathsf{T}$. Delete $\mathsf{T} \setminus v$ from G.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

The only remaining cycles pass through v.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

Find an optimal cut set for paths from $N(v)$ to $N(v)$.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

When is this cut set small enough?

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $\mathsf{N}(v)$ to $\mathsf{N}(v)$ is small.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $\mathsf{N}(v)$ to $\mathsf{N}(v)$ is *not* small...

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

When is this cut set small enough?

When the largest collection of vertex disjoint paths from $\mathsf{N}(v)$ to $\mathsf{N}(v)$ is *not* small... we get a reduction rule.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

When is this cut set small enough?

More than k vertex-disjoint paths from $N(v)$ to $N(v)$
$\rightarrow$ v belongs to *any* feedback vertex set ($k + 1$-flower) of size at most k.

Given a high-degree vertex v, finding a small feedback vertex set that does not contain v.

So either v "forced", or we have feedback vertex set of suitable size.
Notice that we need to arrive at either situation in "polynomial time".

Approximate fvs

- There is a factor 2 approximation algorithm for FEEDBACK VERTEX SET. So use this to get T. If $|\mathsf{T}| > 2\mathsf{k}$ return no-instance. Else, we have the desired T.
-

Approximate fvs

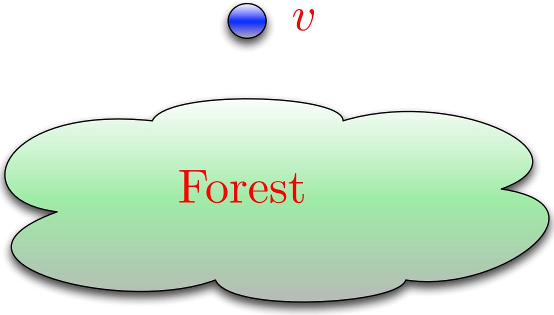
- There is a factor 2 approximation algorithm for FEEDBACK VERTEX SET. So use this to get T. If $|T| > 2k$ return no-instance. Else, we have the desired T.
- We have seen if G has minimum degree 3, then any fvs of size at most k contains one among the first $3k$ vertices of highest degree. Use this to get T of size $3k^2$ or return no-instance.

fvs without v when $v \in \mathsf{T}$.

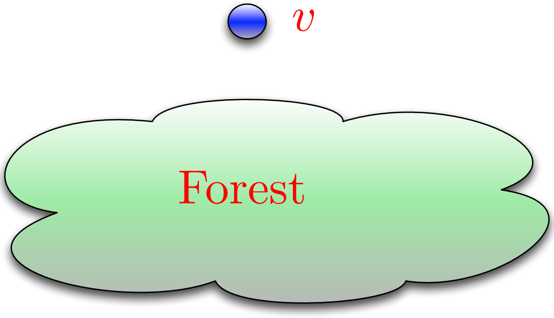
- $Z_v = \mathsf{T} \setminus \{v\} + W$(something more).

fvs without v when $v \in \mathsf{T}$.

- $\mathsf{Z}_v = \mathsf{T} \setminus \{v\} + W$(something more).



fvs without v when $v \in \mathsf{T}$.



W will be a fvs for Forest $+ v$.

- Check whether there is a $k+1$-flower containing v in Forest $+ v$ (if yes then we have reduction rule). (How to find?)
-
-

- Check whether there is a $k+1$-flower containing v in Forest $+ v$ (if yes then we have reduction rule). (How to find?)
- Else, we can show that there is fvs for Forest $+ v$ of size at most $2k$.
-

Book – Gallai Theorem

Theorem (Gallai)

Given a simple graph G, *a set* $\mathsf{T} \subseteq \mathsf{V}(\mathsf{G})$ *and an integer* s, *one can in polynomial time find either*

1. *a family of* $\mathsf{s}+1$ *pairwise vertex-disjoint* T*-paths, or*
2. *a set* B *of at most* $2\mathsf{s}$ *vertices, such that in* $\mathsf{G} \setminus \mathsf{B}$ *no connected component contains more than one vertex of* T*.*

What did we show.

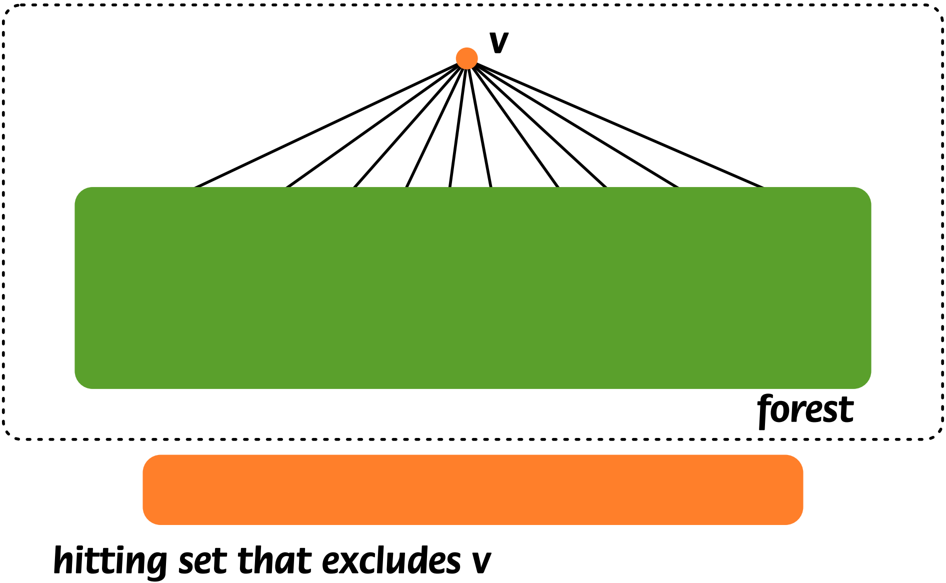
- For every vertex v either there is a $k+1$-flower passing through v or there is a Z_v of size at most $4k$ that does not include v and is a fvs of G.
-
-

What did we show.

- For every vertex v either there is a $k+1$-flower passing through v or there is a Z_v of size at most $4k$ that does not include v and is a fvs of G.
- In the first case we apply Flower Rule.
-

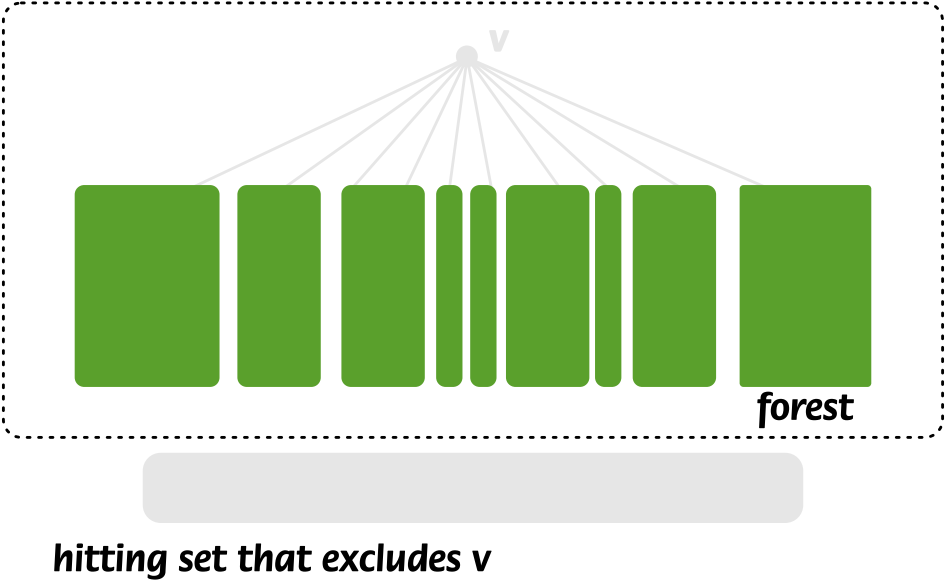
What did we show.

- For every vertex v either there is a $k+1$-flower passing through v or there is a Z_v of size at most $4k$ that does not include v and is a fvs of G.
- In the first case we apply Flower Rule.
- Assume that the first case does not happen, so we have Z_v of size at most $4k$ for every vertex $v \in V(G)$.

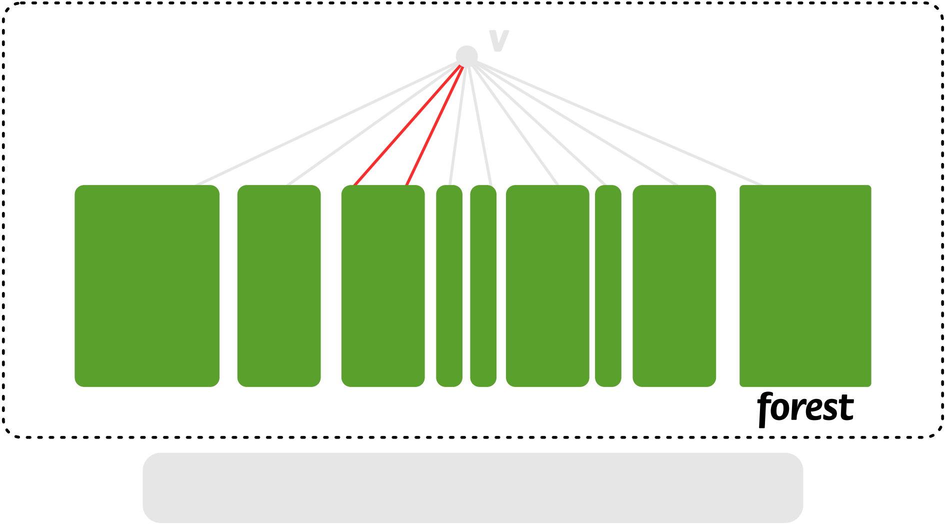

v
forest
hitting set that excludes v

Focussing on the green Part

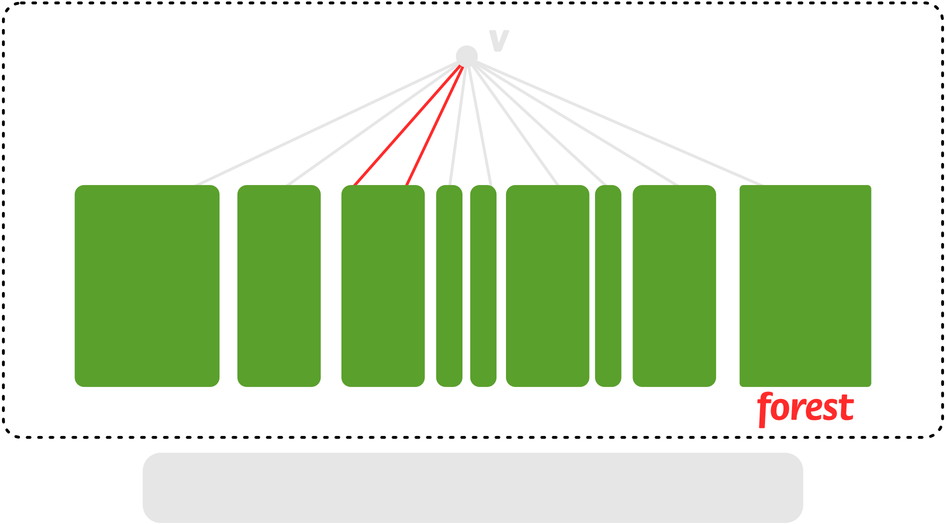
Consider the connected components of $\mathsf{V}(\mathsf{G}) \setminus (\mathsf{Z}_v \cup \{v\})$.


v
forest
hitting set that excludes v

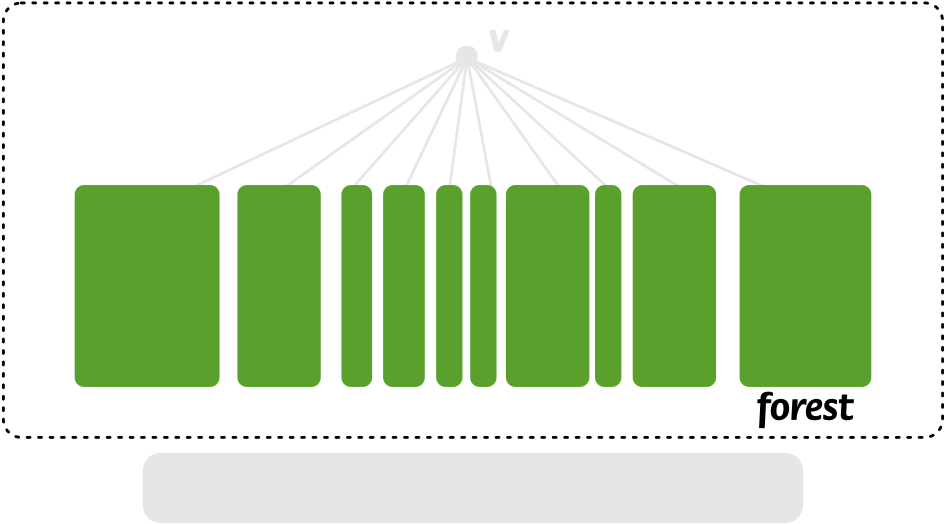
Could v have two neighbor in a connected components of $V(G) \setminus (Z_v \cup \{v\})$?



hitting set that excludes v

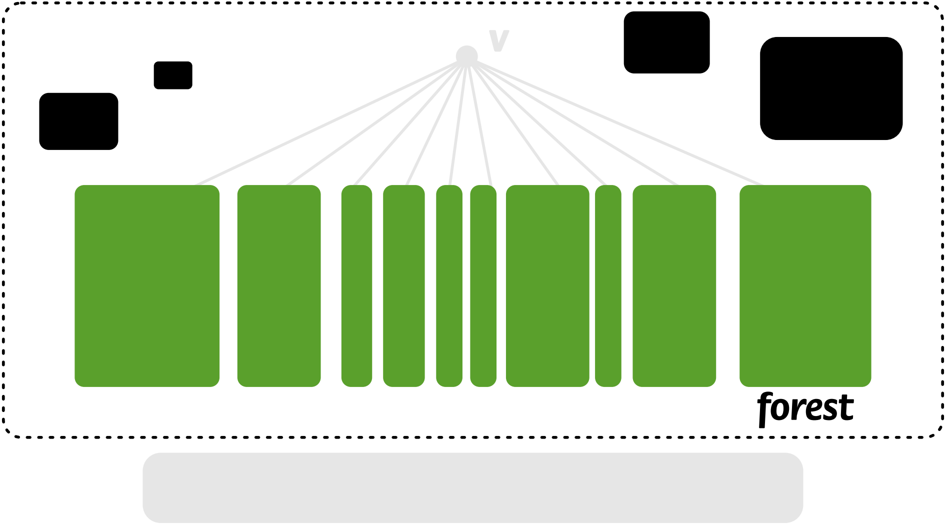


hitting set that excludes v



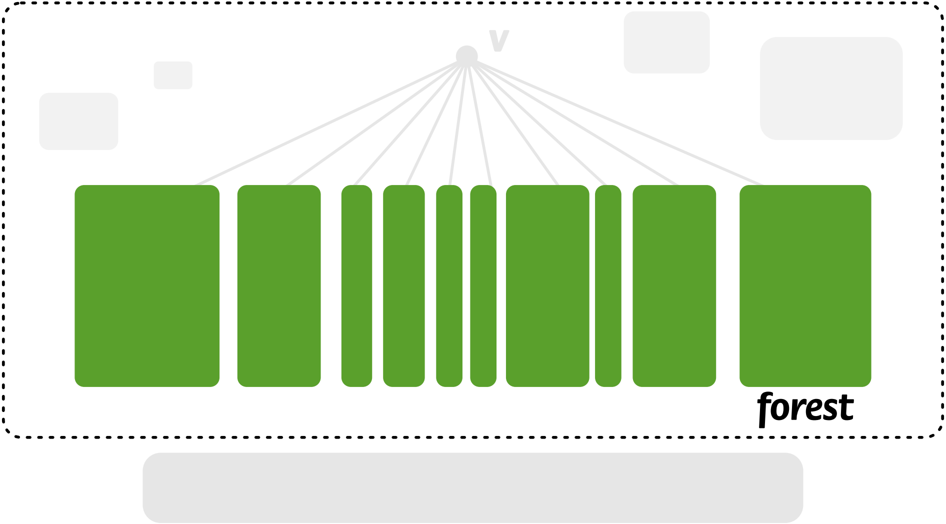
hitting set that excludes v

There could be components in $\mathsf{V}(\mathsf{G}) \setminus (\mathsf{Z}_v \cup \{v\})$ that do not see any neighbor of v. Important, for us is that any component contains at most one neighbor of v and we will focus on them.



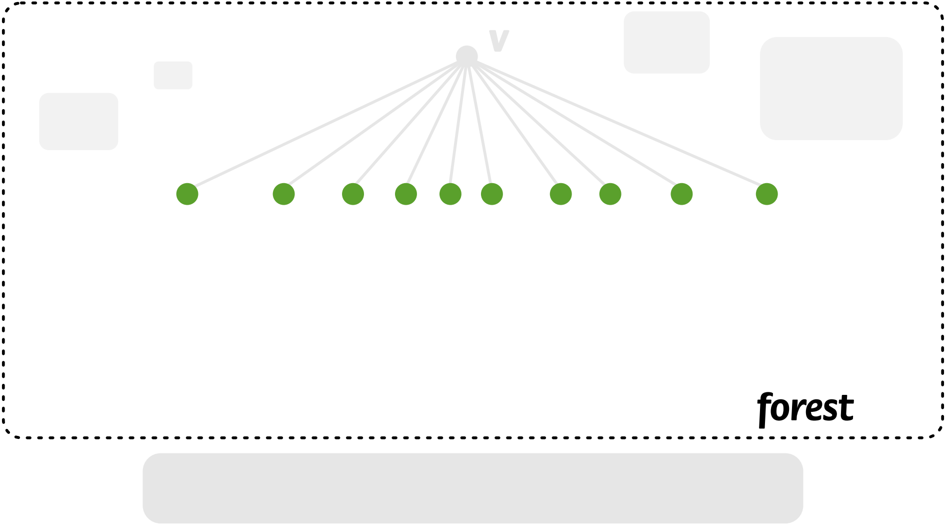
hitting set that excludes v

To bound the degree of v or to delete an edge incident to v we only focus on those components that contain some (exactly one) neighbor of v.



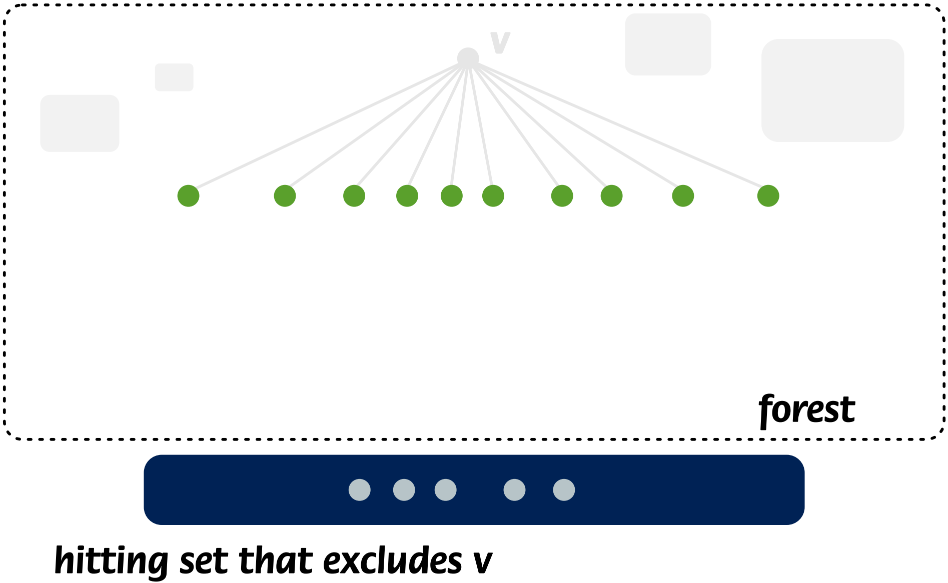
hitting set that excludes v

To apply 2-expansion lemma we need a bipartite graph. In one part (say B) we will have a vertex for every component in $V(G) \setminus (Z_v \cup \{v\})$ that contains a neighbor of v.



hitting set that excludes v

To apply 2-expansion lemma we need a bipartite graph. In one part (say B) we will have a vertex for every component in $V(G) \setminus (Z_v \cup \{v\})$ that contains a neighbor of v. The other part A will be Z_v.


v
forest
hitting set that excludes v

- So we have A and B. We put an edge between a vertex x in A and a vertex w in B, if x is adjacent to some vertex in the component represented by the vertex w. Essentially, we have obtained this bipartite graph by contracting the components.
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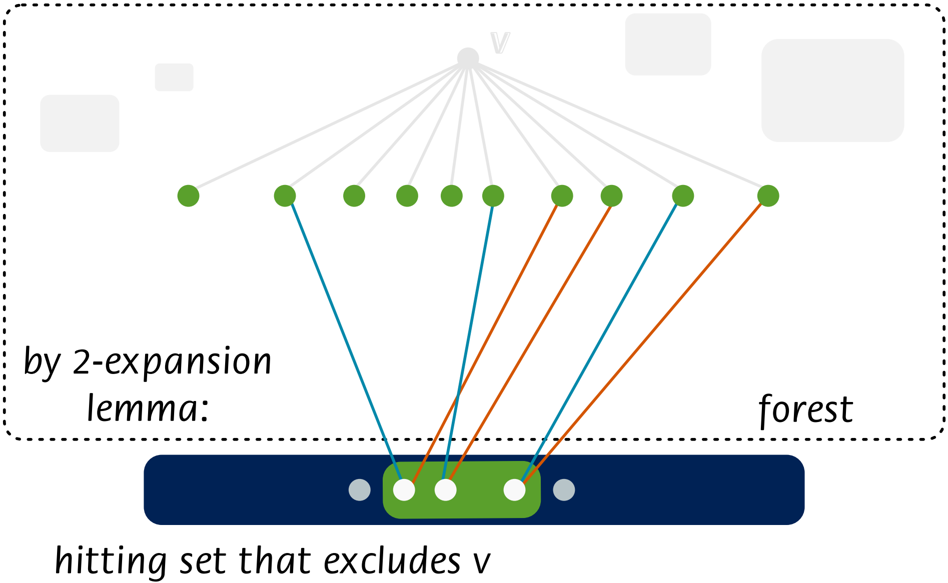
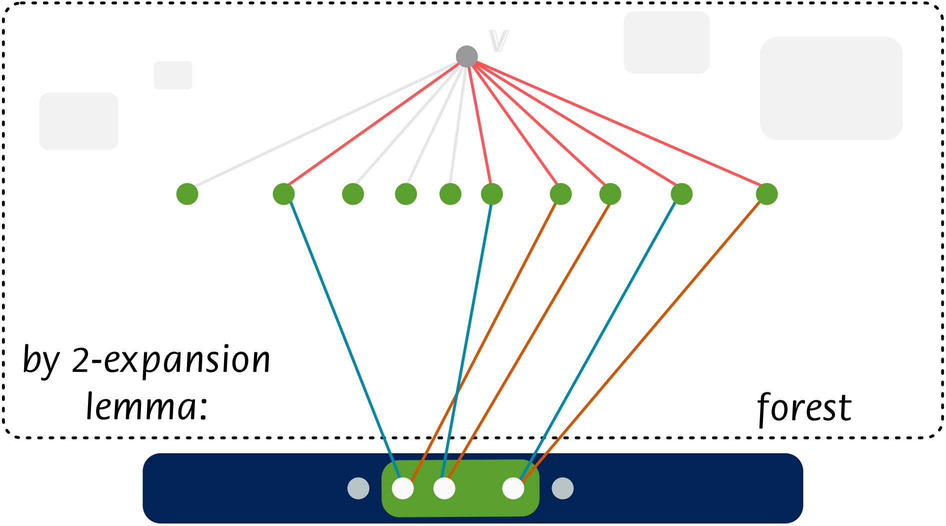
- So we have A and B. We put an edge between a vertex x in A and a vertex w in B, if x is adjacent to some vertex in the component represented by the vertex w. Essentially, we have obtained this bipartite graph by contracting the components.
- If $|B| < 2|A| \leqslant 8k$ then v already has its degree bounded by $8k$. So assume that

$$|B| > 2|A|$$

Now by expansion lemma (applied with $q = 2$) we have that there exist nonempty vertex sets $X \subseteq A$ and $Y \subseteq B$ such that

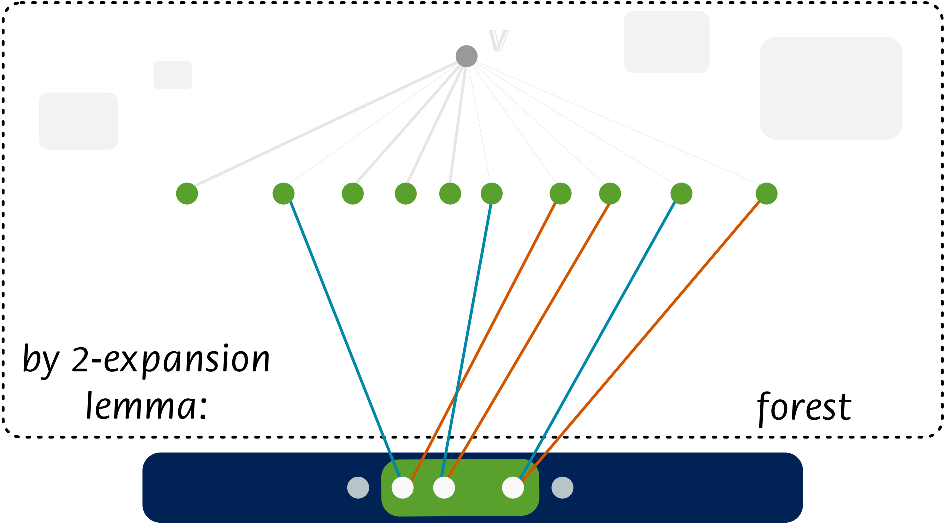
- there is a 2-expansion of X into Y, and
- no vertex in Y has a neighbor outside X, that is, $N(Y) \subseteq X$.


v
by 2-expansion
lemma:
forest
hitting set that excludes v



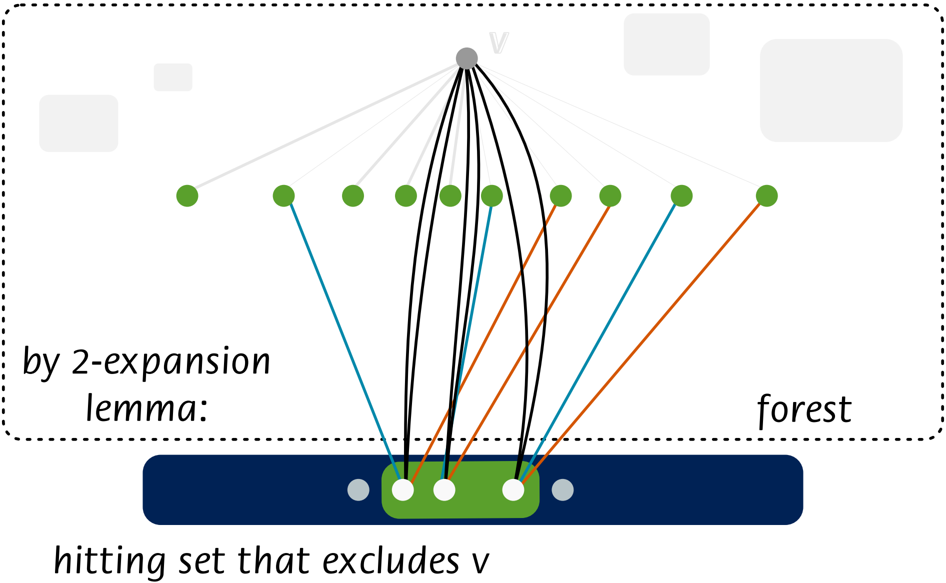
hitting set that excludes v

So the reduction rule is:



hitting set that excludes v

. . . and add the following edges if already not present.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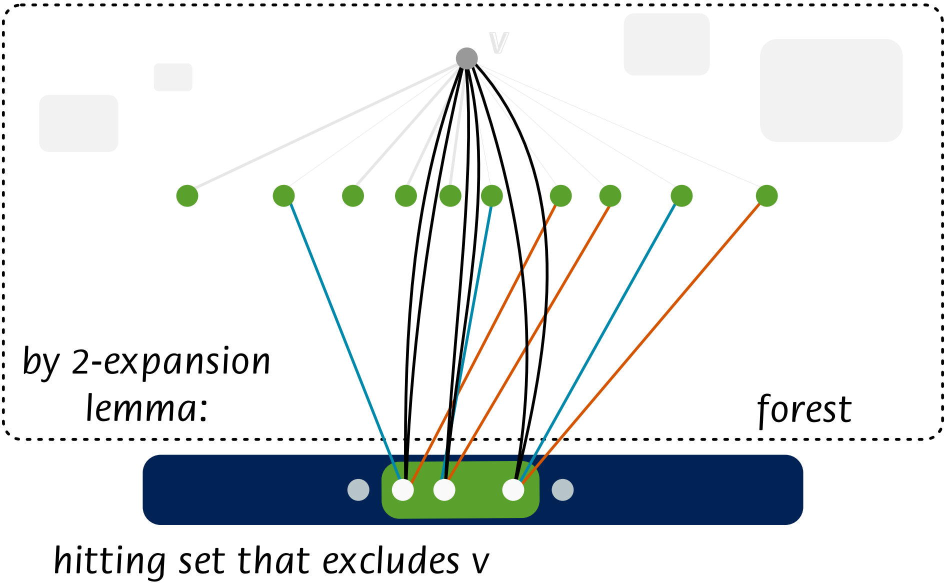
Let us argue correctness!

The Forward Direction

The Forward Direction

$$\text{FVS} \leqslant \mathsf{k} \text{ in } \mathsf{G} \Rightarrow \text{FVS} \leqslant \mathsf{k} \text{ in } \mathsf{H}$$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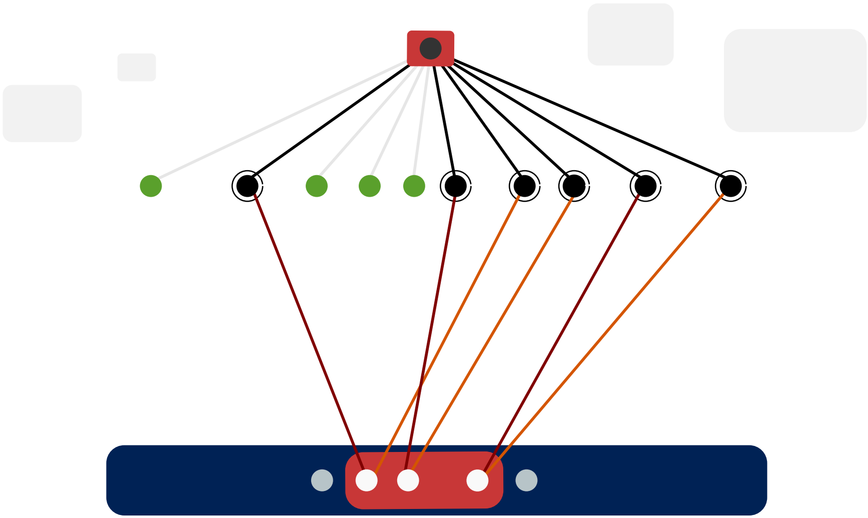
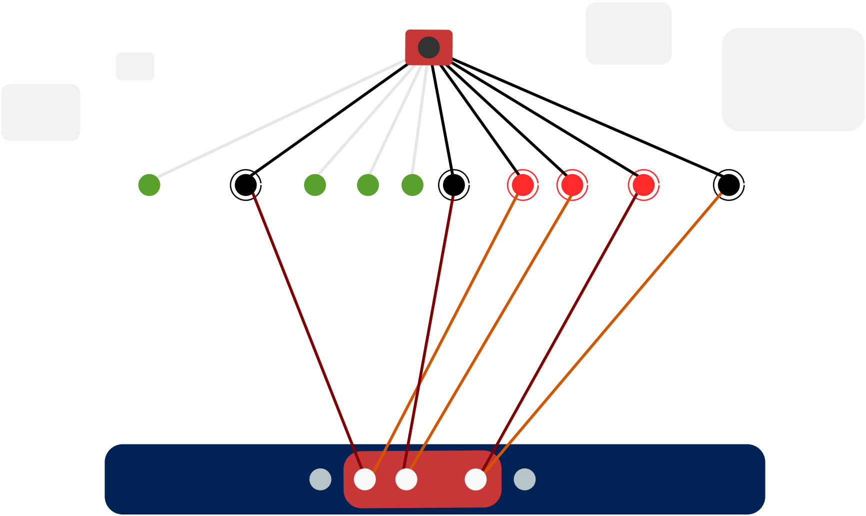
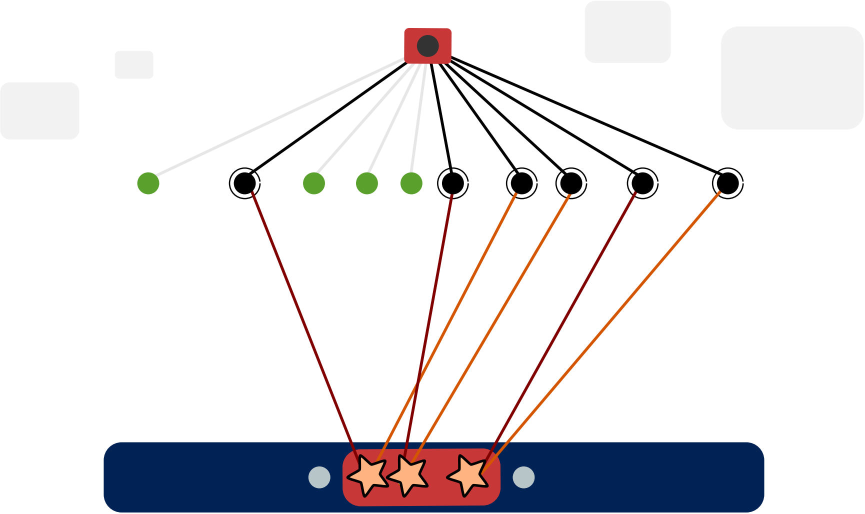
If G has a FVS that either contains v or all of X, we are in good shape.

Consider now a FVS that:

- Does not contain v,
- and omits at least one vertex of X.







Notice that this does not lead to a larger FVS:

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For every vertex v in X that a FVS of G leaves out,

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For every vertex v in X that a FVS of G leaves out,

it must pick a vertex u that kills no more than all of X.

The Reverse Direction

The Reverse Direction

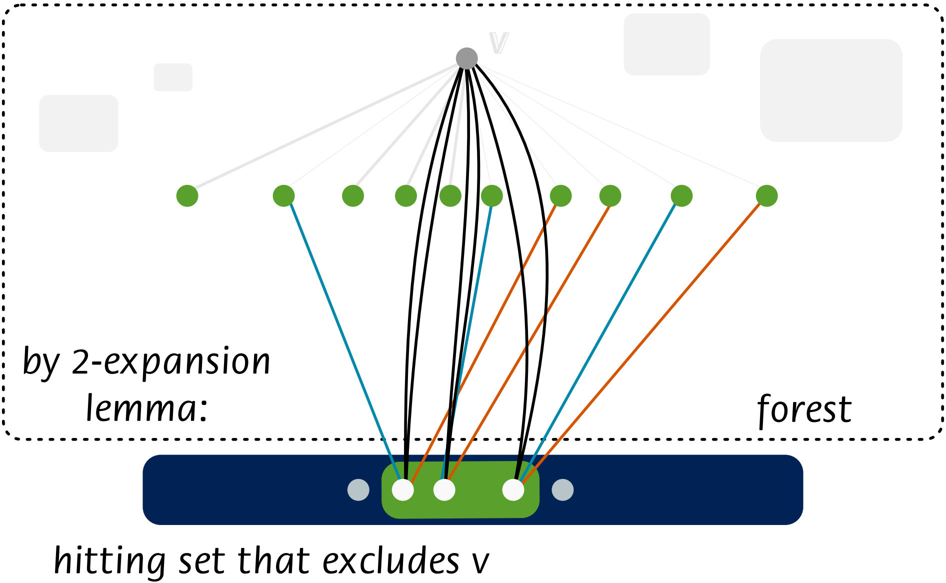
$$\text{FVS} \leqslant \mathsf{k} \text{ in } \mathsf{G} \Leftarrow \text{FVS} \leqslant \mathsf{k} \text{ in } \mathsf{H}$$

The Reverse Direction

The Reverse Direction

$$\text{FVS} \leqslant k \text{ in } G \Leftarrow \text{FVS} \leqslant k \text{ in } H$$

If FVS in H contains v then the same works for G also as $G \setminus \{v\}$ is isomorphic to $H \setminus \{v\}$. So assume that FVS in H does not contain v.


v
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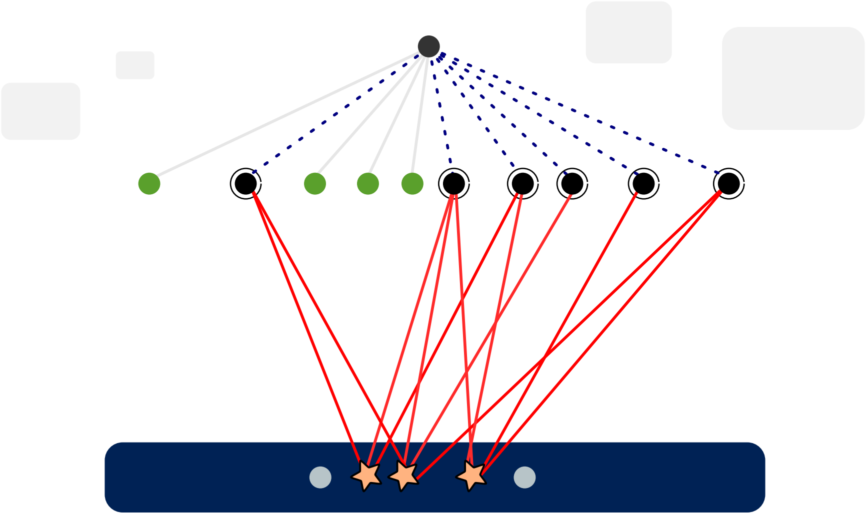
Let W be a FVS of H, the Only Danger for W to be a FVS of G:

Cycles that:

- pass through v,
- non-neighbors of v in H (neighbors in G, however)
- and do not pass through X.

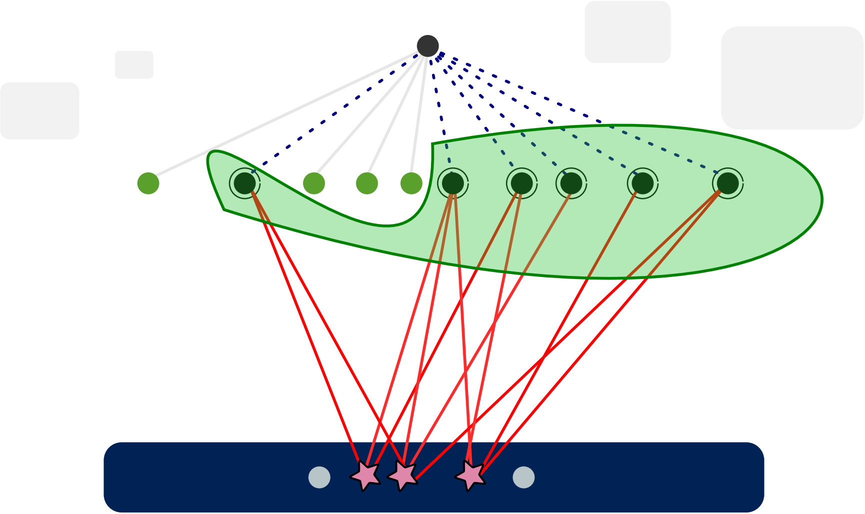


Let W be a FVS of H, the Only Danger for W to be a FVS of G:

Cycles that:

- pass through v,
- non-neighbors of v in H (neighbors in G, however)
- and do not pass through X.

However recall that $\mathsf{N(Y)} \subseteq \mathsf{X}$.



Wrapping Up

- A priori it is not obvious that previous Reduction Rule actually makes some simplification of the graph, since it *substitutes some set of edges with some other set of double edges!*
-

Wrapping Up

- A priori it is not obvious that previous Reduction Rule actually makes some simplification of the graph, since it *substitutes some set of edges with some other set of double edges!*
- We need to formally prove that the reduction rules cannot be applied infinitely, or superpolynomially many times.

Final Result

Theorem

FEEDBACK VERTEX SET *admits a kernel with at most* $O(k^2)$ *vertices and* $O(k^2)$ *edges.*

Thank You.

Slices Courtesy: Neeldhara Misra and Saket Saurabh.