Parameterized Algorithms

Lecture 6: Algebraic Methods

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Parameterized Algorithms via Set Systems, Polynomials etc.

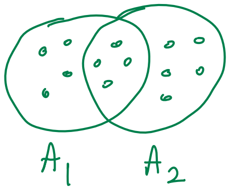
Inclusion-Exclusion Principle

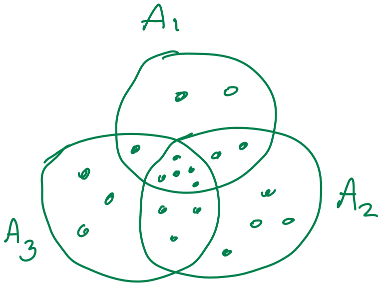
Inclusion-Exclusion

Theorem

Let $A_1, A_2, \ldots, A_k$ be subsets of a universe U, and let $B_i = U \setminus A_i$. Then

$$|\bigcap_{i\in[k]} A_i| = \sum_{X\subseteq[k]} (-1)^{|X|} |\cap_{j\in X} B_j|$$





Unweighted Steiner Tree

Unweighted Steiner Tree: Given a graph G in n vertices, a subset K of k terminals, find a subgraph(tree) on at most ℓ edges that connects all the terminals.[1]

Theorem

Unweighted Steiner Tree *can be solved in* $2^k \cdot poly(n)$ *time.*

Using Inclusion-Exclusion

[1] $\ell \geq k$, and we can always guess the smallest value of ℓ for which a Steiner Tree exists.

Unweighted Steiner Tree

Intuition:

- We solve the Counting Problem.

  If the number of ℓ-edge subtrees of G containing K is non-zero, then a Steiner Tree on ℓ edges exists.

- Counting trees is hard, so we count an easier object called Branching Walks.
- We count Branching Walks via Inclusion-Exclusion.

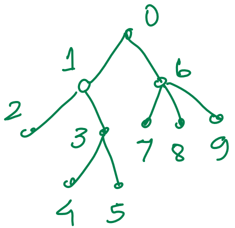
Unweighted Steiner Tree

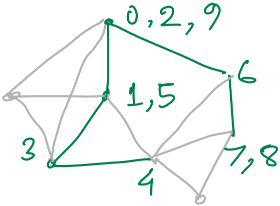
Ordered Rooted Tree : A tree H where vertices have been labeled by $\{0, 2, 3 \ldots, |V(H)| - 1\}$ via a DFS. Alternatively, every internal node of H has an ordering among it's children.

Let $r \in V(H)$ denote the <u>root</u> of H.

Branching Walk : A Homomorphic Image of an ordered rooted tree in G. It is a pair $B = (H, h)$ where H is an ordered rooted tree, and $h : V(H) \to V(G)$ is a map such that if $(x, y) \in E(H)$ then $(h(x), h(y) \in E(G)$.

Let $V(B) = \{h(x) \mid x \in V(H)\}$, and $s = h(r)$ be the <u>root</u> of B.





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Let $V(B) = \{h(x) \mid x \in V(H)\}$, and $s = h(r)$ be the root of B.

Lemma

Fix a terminal $s \in K$ as the root. G contains a Steiner Tree on ℓ edges if and only if there is a Branching Walk $B = (H, h)$ from s such that $K \subseteq V(B)$, and $|E(H)| \leq \ell$.

Call $\ell = |E(H)|$ the length of the Branching Walk.

Unweighted Steiner Tree

Counting Branching Walks from s.

- Universe U = all Branching walks of length ℓ from s
- For each $v \in K$, $A_v = \{B \in U \mid v \in V(B)\}$
- Clearly $|\bigcap_{v \in K} A_v| \neq 0$ if and only if there is a Steiner Tree.
- Sufficient: Given $X \subseteq K$ compute $|\bigcap_{v \in X} B_v|$ where $B_v = U \setminus A_v$.

  $\bigcap_{v \in X} B_v$ is the set of all Branching Walks that avoid X

- They lie in the graph $G - X$,

  so enough to count all Branching Walks from s in the graph $G - X$ of length ℓ.

Lemma

$|\bigcap_{v \in X} B_v|$ can be computed in polynomial time.

Unweighted Steiner Tree

Computing $|\bigcap_{v \in X} B_v|$:

- Let $G' = G - X$. It contains all Branching Walks avoiding X.
- For $u \in V(G')$ and $j \leq \ell$ let $b_j(u)$ denote the number of Branching Walks from u of length j in G'.
- We want the value $b_\ell(s) = |\bigcap_{v \in X} B_v|$, assuming that $s \in V(G')$.
- Dynamic Programming:

$$b_j(u) = \begin{cases} 1 & \text{if } j = 0 \\ \sum_{w \in N_{G'}(a)} \sum_{j_1 + j_2 = j-1} \quad b_{j_1}(u) b_{j_2}(w) & \text{otherwise} \end{cases}$$

Unweighted Steiner Tree

Counting Steiner Trees

- Once we have the numbers $|\bigcap_{v\in X} B_v|$ for every $X \subseteq K$, we can compute the number of Steiner Trees via the Inclusion-Exclusion formula

$$|\bigcap_{v\in K} A_v| = \sum_{X\subseteq K} (-1)^{|X|} |\cap_{u\in X} B_u|$$

- Running Time: $2^k \cdot poly(n)$.

This approach can be applied to many other problems such as Hamiltonian Path, Chromatic Number etc.

Multivariate Polynomials: FPT Algorithms

Multivariate Polynomials

- **Finite Field:** A tuple $(\mathbb{F}, +, \star)$ capturing arithmetic in a finite set.
- **Characteristic 2:** For any $a \in \mathbb{F}$ $a + a = 0$.
  Note that $|\mathbb{F}| >> 2$ is possible.
- **Polynomials over $\mathbb{F}$:** coefficients $a_{\dots} \in \mathbb{F}$

$$P(x_1, x_2, \dots, x_n) = \sum_{(c_1, c_2, \dots, c_n) \in (\mathbb{N} \cup \{0\})^n} a_{c_1, c_2, \dots, c_n} x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$$

  degree of $P = \max_{(c_1, c_2, \dots, c_n) | a_{c_1, c_2, \dots, c_n} \neq 0} \sum c_i$ where
- **Identically Zero Polynomial:** $P \equiv 0$ means
  $P(x_1 = b_1, x_2 = b_2, \dots, x_n = b_n) = 0$ for all choices in $\mathbb{F}^n$

Lemma (Schwartz-Zippel)

Let P be a polynomial over a field $\mathbb{F}$ of degree d, and let $S \subseteq \mathbb{F}$. Pick $b_1, b_2, \dots, b_n$ randomly from S. If $P \not\equiv 0$, then $P(b_1, b_2, \dots, b_n) = 0$ with probability at most $d/|S|$.

k-Path

k-Path: Given a graph G and an integer k, decide if G contains a path of length k.

Theorem

There is a randomized FPT algorithm for k-Path *running in time* $2^k \cdot poly(n)$.

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Intuition

- Encode k-walks as monomials of a polynomial
- Ensure the walks "cancel out" (using characteristic 2), hence the polynomial encodes only k-paths
- The polynomial is non-zero means there is a k-path. Test using Schwartz-Zippel Lemma.

Path to Polynomials

- variables $x =< x_1, \ldots, x_m >$ for edges,
  $y =< y_1, \ldots, y_n >$ for vertices.
- Path polynomial (hard to eval)

$$P(x,y) = \sum_{k\text{-Path } R \in G} \Big(\prod_{(v_i, v_{i+1}) \in R} x_{v_i, v_{i+1}} \Big) \cdot \Big(\prod_{v_i \in R} y_{v_i} \Big)$$

- Walk polynomial (easy to eval, but not very useful)

$$P(x,y) = \sum_{k\text{-Walk } W \in G} \Big(\prod_{(v_i, v_{i+1}) \in W} x_{v_i, v_{i+1}} \Big) \cdot \Big(\prod_{v_i \in W} y_{v_i} \Big)$$

- **Labeled Walk Polynomial.**
  - vertex variable set $y = \{y_{v,i} \mid v \in V(G), i \in [k]\}$
  - For a bijective function $\ell : [k] \to [k]$ and a k-Walk W we have the monomial $mon(W, \ell) = (\prod_{(v_i, v_{i+1}) \in W} x_{v_i, v_{i+1}}) \cdot (\prod_{v_i \in W} y_{v_i, \ell(i)})$

$$P(x,y) = \sum_{\text{Walks } W} \ \sum_{\text{bijection } \ell} mon(W, \ell)$$

Path to polynomials

Lemma

Over a field of characteristic 2,

$$P(x,y) \equiv \sum_{Paths\ R} \sum_{bijection\ \ell} mon(R,\ell)$$

- Any k-Walk W corresponds to a number of labeled walks, one for each bijection $\ell : [k] \to [k]$.
- For a k-Path R, every bijection ℓ gives a distinct monomial.
- However for a walk W, for every bijection ℓ there is another bijection ℓ' that produces the same monomial, and they cancel out.
  - For a walk W where a vertex v repeats at pos a and b
  - Given $\ell : [k] \to [k]$ define

$$\ell'(i) = \begin{cases} \ell(b) & i = a \\ \ell(a) & i = b \\ \ell(i) & \text{otherwise} \end{cases}$$

Path to polynomials

Lemma

Over a field of characteristic 2,

$$P(x,y) \equiv \sum_{Paths\ R} \sum_{bijection\ \ell} mon(R,\ell)$$

Corollary

The polynomial $P(x,y)$ is non-zero over fields of characteristic 2 if and only if G contains a k-path.

- We test if $P \equiv 0$ using the Schwartz-Zippel Lemma
- We randomly pick an assignment of the variables from $\mathbb{F}$ and then evaluate P.
- Evaluating P will require an algorithm based on Inclusion-Exclusion.

Evaluating $P(x, y)$

Theorem (Weighted Inclusion Exclusion)

Let $A_1, A_2, \ldots, A_k$ *be subsets of a universe* U, *and let* $B_i = U \setminus A_i$. *Let* $w : U \to \mathbb{R}$ *be a weight function Then*

$$w\Big(\bigcap_{i \in [k]} A_i\Big) = \sum_{X \subseteq [k]} (-1)^{|X|} w(\cap_{j \in X} B_j)$$

Evaluating $P(x, y)$

Fix a walk W

- Universe U = all functions $[k] \to [k]$
- for $\ell \in U$, define $w(\ell) = mon(W, \ell)$
- For each $i \in [k]$, $A_i = \{\ell \in U \mid \ell^{-1}(i) \neq \emptyset\}$
- Then $w(\cap_{i \in [k]} A_i) = \sum_{\text{bijection } \ell} \; mon(W, \ell)$
- $w(\bigcap_{i \in [k]} A_i) = \sum_{X \subseteq [k]} w(\cap_{j \in X} B_j)$,
- and $\sum_{X \subseteq [k]} w(\cap_{j \in X} B_j) = \sum_{X \subseteq [k]} \sum_{\ell : [k] \to [k] \setminus X} mon(W, \ell)$,

Therefore,

$$\begin{aligned} P(x, y) &= \sum_{\text{Walks } W} \; \sum_{\text{bijection } \ell} mon(W, \ell) \\ &= \sum_{\text{Walks } W} \; \sum_{X \subseteq [k]} \; \sum_{\ell : [k] \to [k] \setminus X} mon(W, \ell) \end{aligned}$$

Evaluating $P(x,y)$

$$P(x,y) = \sum_{X \subseteq [k]} \sum_{\text{Walks } W} \sum_{\ell:[k]\to[k]\setminus X} mon(W,\ell)$$

- fixing $X \subseteq [k]$ and let $Y = [k] \setminus X$ we obtain a polynomial

$$P_Y(x,y) = \sum_{\text{Walks } W} \sum_{\ell:[k]\to Y} mon(W,\ell)$$

- To evaluate $P_Y(x,y)$ we use Dynamic Programming.
- For $d \le k$, and vertex v

$$T[v,d] = \sum_{\text{Walk } W: v=v_1 v_2 \ldots v_d} \sum_{\ell:[d]\to Y} (\prod_{e\in W} x_e)(\prod v_i \in W y_{v_i,\ell(i)})$$

We want the value $T[v,k]$ for all vertices $v \in V(G)$.

Evaluating $P(x, y)$

$$T[v,d] = \begin{cases} \sum_{i \in Y} y_{v,i} & d = 1 \\ \sum_{i \in Y} y_{v,i} \sum_{(v,w) \in E(G)} x_{v,w} \cdot T[w, d-1] & \text{otherwise} \end{cases}$$

Once we have computed this table,

$$P_Y(x,y) = \sum_{v \in V(G)} T[v,k]$$

Then over all $Y \subseteq [k]$

$$P(x,y) = \sum_{Y \subseteq [k]} P_Y(x,y)$$

Summary: k-Path via Polynomials

Theorem

There is a randomized FPT algorithm for k-Path running in time $2^k \cdot poly(n)$.

Thank you