UNIVERSITÄT DES SAARLANDES

max planck institut informatik

# Sublinear Algorithms

## Lecture 01: Introduction & Streaming I

Karl Bringmann

May 07, 2020

European Research Council
Established by the European Commission

# Etiquette

- These **slides** will be available on the course website
- The lecture is **recorded** and a video will be made available to all participants
- We start every meeting with everyone’s (but mine) **mics and videos off**, to avoid noise and save bandwidth

**Questions:**

- can always be asked in the **chat**, I will keep an eye on the chat window
- during **breaks** the recording is paused, so it is safe to turn on your video and ask questions
- during recording, you can also unmute and ask questions, but be aware that you are recorded and the video will be made available to all participants

# Organization

**Advanced Lecture, 2+1, 5CP**

**Lecture:** Karl Bringmann and Vasileios Nakos
every Thursday 16-18 (on holidays: move to Monday 16-18)
via Zoom + video download

**Tutorial:** Nick Fischer
every second Monday 16-18
via Zoom

**Requirements:** basic algorithms lecture, e.g., Grundzüge von Algorithmen und Datenstrukturen

**Exam:** oral exam
admittance by $\geq$50% of points on 4 exercise sheets

https://lists.mpi-inf.mpg.de/listinfo/sublinear

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer20/sublinear-algorithms/

# Course Overview

**NP:** $O(2^n)$ $O(2^{\sqrt{n}})$ $O(n^n)$

**P:** $O(n^{100})$ $O(n^2)$ $O(n \log n)$ $O(n)$

**Space $\boldsymbol{o(n)}$?**

**#Measurements $\boldsymbol{o(n)}$?**

**Time $\boldsymbol{o(n)}$?**

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**Streaming Algorithms:**

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Data stream $x_1, x_2, \ldots, x_n$

Make one pass over the stream

Working memory $o(n)/O(\log n)$

$\approx$ low-space data structures

Typical problems:

- Compute number of distinct $x_i$'s
- Compute the majority element (if exists)
- Compute all numbers that appear $\geq \varepsilon n$ times

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**Space $o(n)$?**

**#Measurements $o(n)$?**

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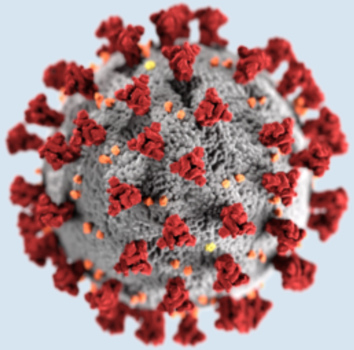
**Randomized Trials:**

Estimate the infected population
by testing random individuals



**Combinatorial Group Testing:**

Mix samples of a group of individuals → test tells us whether at least one individual is positive

Find *all* positive individuals using $o(n)$ group tests

**Medical Imaging:**

Reconstruct a sparse vector
from few Fourier measurements

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**Space $\boldsymbol{o(n)}$?**

**#Measurements $\boldsymbol{o(n)}$?**

**Time $\boldsymbol{o(n)}$?**

**Property Testing:**

Really sublinear time $o(n)$!

"What can we find out about $x_1, x_2, \dots, x_n$

using $o(n)$ random accesses?"

Typical problems:

Is $x_1, x_2, \dots, x_n$ monotone or *far* from monotone?

Is a graph 2-colorable or *far* from 2-colorable?

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**Space $\boldsymbol{o(n)}$?**

**#Measurements $\boldsymbol{o(n)}$?**

**Time $\boldsymbol{o(n)}$?**

**Course Outline:**

3x Streaming (Space)

4x Vector Reconstruction (Measurements)

2x Property Testing (Time)

2x Applications

# Outline

1) Course Overview
2) **Basic Probability Theory**
3) Morris' Counter

Questions?

5min Break?

# Basic Probability Theory

Course Website → Material → A Primer to Randomness

**Random Variable:**

$X$ is a random coin flip

$\mathbb{P}[X = 0] = \mathbb{P}[X = 1] = \frac{1}{2}$

$\mathbb{E}[X] = \frac{1}{2}$

$X = X_1 + X_2$, where
$X_1, X_2$ are random coin flips

$\mathbb{P}[X = 0] = \mathbb{P}[X = 2] = \frac{1}{4}$
and $\mathbb{P}[X = 1] = \frac{1}{2}$

$\mathbb{E}[X] = 1$

**Expectation:**

$$\mathbb{E}[X] = \sum_n n \cdot \mathbb{P}[X = n]$$

**Linearity of Expectation:**

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

**Event:**

$X = 0$

$X \leq 1$

$X$ is even

**Union Bound:**

$$\mathbb{P}[A \text{ or } B] \leq \mathbb{P}[A] + \mathbb{P}[B]$$

# Concentration Inequalities

**Markov:** $$\mathbb{P}[X \geq t] \leq \frac{\mathbb{E}[X]}{t}$$ For any $t > 0$, assuming $X \geq 0$

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For any $t > 0$, assuming $X \geq 0$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

For any $t > 0$

*Variance* $\mathrm{Var}[X] = \underline{\mathbb{E}[X^2] - \mathbb{E}[X]^2}$

# Concentration Inequalities

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For any $t > 0$

*Variance* $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

**Chernoff:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2\exp\left(-\frac{2t^2}{n}\right)$

For any $t > 0$, assuming $X = X_1 + \cdots + X_n$

with *independent* $X_1, \dots, X_n \in \{0,1\}$

for any values $x_1, \dots, x_n$:

$\mathbb{P}[X_1 = x_1 \textit{ and } \dots \textit{ and } X_n = x_n]$

$= \mathbb{P}[X_1 = x_1] \cdot \ldots \cdot \mathbb{P}[X_n = x_n]$

# Outline

1) Course Overview
2) Basic Probability Theory
3) **Morris' Counter**

Questions?

5min Break?

# Counting

*Most simple streaming problem*

monitor a sequence of events, maintain a **counter** of the number of events

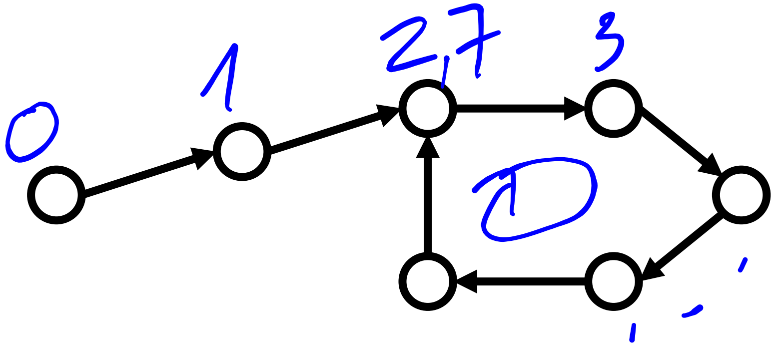
**Data structure problem:**

maintain a number $n$

**update():** increment $n$ by 1

**query():** output $n$

initially $n = 0$




**Solution:** Standard Counter

store $n$ using $\lceil \log n \rceil = O(\log n)$ bits

**This is optimal:** with $< \log n$ bits...

... we must make an error

$\rightarrow$ need **approximation**

... we run into infinite loops

$\rightarrow$ need **randomization**

# Approximate Counting

Goal: $O(\log \log n)$ space

monitor a sequence of events, maintain an **approximate counter** of the number of events

**Data structure problem:**

maintain a number $n$

**update():** increment $n$ by 1

**query():** output $\tilde{n}$ with

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \delta$$

In other words: $(1-\varepsilon)n < \tilde{n} < (1+\varepsilon)n$
with probability at least $1 - \delta$

$\varepsilon, \delta \in (0,1)$ are parameters given to the algorithm upfront

**Solution:** Morris' Counter (1978)

1) Initialize $X = 0$

2) On update(): Increment $X$ with probability $2^{-X}$

3) On query(): output $\tilde{n} = 2^X - 1$

Intuition: store integer $X \approx \log n$

When to increment?

Increment with probability $\frac{1}{n} \approx 2^{-X}$

# Approximate Counting

**Lem:** Morris' Counter is an **unbiased estimator** of $n$, that is, $\mathbb{E}[\tilde{n}] = n$.

Morris' Counter

1) Initialize $X = 0$
2) On update(): Increment $X$ with probability $2^{-X}$
3) On query(): output $\tilde{n} = 2^X - 1$

**Proof:** Consider one update in isolation

Let $X, X'$ be the counter before/after the update

$$\mathbb{E}[2^{X'} | X] = \frac{1}{2^X} \cdot 2^{X+1} + \left(1 - \frac{1}{2^X}\right) \cdot 2^X$$

$$= 2^X + 1$$

express expectation of $2^{X'}$ in terms of $X$

$$\mathbb{E}[2^{X_n}] = \mathbb{E}[2^{X_{n-1}}] + 1$$

Thus, inductively after $n$ updates we have: $\mathbb{E}[2^X] = n + 1$

# Approximate Counting

**Lem:** Morris‘ Counter is an **unbiased estimator** of $n$, that is, $\mathbb{E}[\tilde{n}] = n$.

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\mathbb{E}[\tilde{n}^2] - n^2}{\varepsilon^2 n^2}$$

**Lem:** We have $\mathbb{E}[\tilde{n}^2] = \frac{3}{2}n^2 - \frac{1}{2}n$.

Morris‘ Counter

1) Initialize $X = 0$
2) On update(): Increment $X$ with probability $2^{-X}$
3) On query(): output $\tilde{n} = 2^X - 1$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[4^{X'} | X] = \frac{1}{2^X} \cdot 4^{X+1} + \left(1 - \frac{1}{2^X}\right) \cdot 4^X$$
$$= 4^X + 3 \cdot 2^X$$

$$\mathbb{E}[\tilde{n}^2] = \mathbb{E}[(2^X - 1)^2] = \mathbb{E}[4^X] - 2\mathbb{E}[2^X] + 1$$

$$\mathbb{E}[4^{X_n}] = \mathbb{E}[4^{X_{n-1}}] + 3 \cdot \mathbb{E}[2^{X_{n-1}}] = \mathbb{E}[4^{X_{n-1}}] + 3n = 3 \cdot \binom{n+1}{2}$$

# Approximate Counting

**Lem:** Morris' Counter is an **unbiased estimator** of $n$, that is, $\mathbb{E}[\tilde{n}] = n$.

**Lem:** We have $\mathbb{E}[\tilde{n}^2] \overset{\leq}{=} \frac{3}{2}n^2 - \cancel{\frac{1}{2}n}$.

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\frac{3}{2}n^2 - n^2}{\varepsilon^2 n^2} \leq \boxed{\frac{1}{2\varepsilon^2}} > 1$$

No approximation guarantee for $\varepsilon \leq 0.7$ !

Morris' Counter

1) Initialize $X = 0$

2) On update(): Increment $X$ with probability $2^{-X}$

3) On query(): output $\tilde{n} = 2^X - 1$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

Questions?

5min Break?

# Boosting via Chebyshev

**Morris+:** *average over $s$ runs of Morris*

1) Run $s$ independent copies of Morris' Counter
2) Let $\tilde{n}_1, \dots, \tilde{n}_s$ be their estimates
3) output $\tilde{n}^+ = \frac{1}{s} \cdot (\tilde{n}_1 + \cdots + \tilde{n}_s)$

**Lem:** For $s \geq 1/(2\varepsilon^2\delta)$ we have

$$\mathbb{P}[|\tilde{n}^+ - n| \geq \varepsilon n] \leq \delta$$

Morris' Counter computes estimate $\tilde{n}$ s.t.

$$\mathbb{E}[\tilde{n}] = n \text{ (unbiased estimator)}$$

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\mathrm{Var}[\tilde{n}]}{\varepsilon^2 n^2} \leq \frac{1}{2\varepsilon^2}$$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

**Proof:** Morris+ is an unbiased estimator:

$$\mathbb{E}[\tilde{n}^+] = \frac{1}{s} \cdot (\mathbb{E}[\tilde{n}_1] + \cdots + \mathbb{E}[\tilde{n}_s]) = n$$

by linearity of expectation

# Boosting via Chebyshev

**Morris+:** *average over $s$ runs of Morris*

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**Lem:** For $s \geq 1/(2\varepsilon^2\delta)$ we have

$$\mathbb{P}[|\tilde{n}^+ - n| \geq \varepsilon n] \leq \delta$$

Morris' Counter computes estimate $\tilde{n}$ s.t.

$\mathbb{E}[\tilde{n}] = n$ (unbiased estimator)

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\mathrm{Var}[\tilde{n}]}{\varepsilon^2 n^2} \leq \frac{1}{2\varepsilon^2}$$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

$$\mathrm{Var}[X] = \underline{\mathbb{E}[X^2] - \mathbb{E}[X]^2}$$

**Proof:** Morris+ is an unbiased estimator

Variance of Morris+ is $\frac{\mathrm{Var}[\tilde{n}]}{s}$:

$$\mathrm{Var}[\tilde{n}^+] = \frac{1}{s^2} \cdot (\mathrm{Var}[\tilde{n}_1] + \cdots + \mathrm{Var}[\tilde{n}_s]) = \frac{\mathrm{Var}[\tilde{n}]}{s}$$

$\mathrm{Var}[\tilde{n}]$

1) $\mathrm{Var}[\alpha X] = \alpha^2 \mathrm{Var}[X]$ 2) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ for independent $X, Y$

# Boosting via Chebyshev

**Morris+:** *average over s runs of Morris*

1) Run $s$ independent copies of Morris‘ Counter

2) Let $\tilde{n}_1, \ldots, \tilde{n}_s$ be their estimates

3) output $\tilde{n}^+ = \frac{1}{s} \cdot (\tilde{n}_1 + \cdots + \tilde{n}_s)$

**Lem:** For $s \geq 1/(2\varepsilon^2\delta)$ we have

$$\mathbb{P}[|\tilde{n}^+ - n| \geq \varepsilon n] \leq \delta$$

Morris‘ Counter computes estimate $\tilde{n}$ s.t.

$\mathbb{E}[\tilde{n}] = n$ (unbiased estimator)

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\mathrm{Var}[\tilde{n}]}{\varepsilon^2 n^2} \leq \frac{1}{2\varepsilon^2}$$

**Chebyshev:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

**Proof:** Morris+ is an unbiased estimator

Variance of Morris+ is $\frac{\mathrm{Var}[\tilde{n}]}{s}$

So by Chebyshev:

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \frac{\mathrm{Var}[\boxed{\tilde{n}^+}]}{\varepsilon^2 n^2} \leq \frac{1}{2s\varepsilon^2} \leq \delta$$

# Approximate Counting

Goal: $O(\log\log n)$ space

monitor a sequence of events, maintain an **approximate counter** of the number of events

**Data structure problem:**

maintain a number $n$

**update():** increment $n$ by 1

**query():** output $\tilde{n}$ with

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \delta$$

**Solution:** Morris‘ Counter (1978)

1) Initialize $X = 0$

2) On update(): Increment $X$ with probability $2^{-X}$

3) On query(): output $\tilde{n} = 2^X - 1$

**Morris+:** *average over* $s = \lceil 1/(2\varepsilon^2\delta) \rceil$ *runs of Morris*

once a counter reaches $X = \boxed{\log\left(\frac{ns}{\delta}\right)}$,

it is incremented with probability at most $\boxed{\frac{\delta}{ns}}$, so

by union bound no such counter is ever incremented

**Space usage:** $O\left(\underbrace{\boxed{\frac{1}{\varepsilon^2\delta}}}_{s} \log\log\left(\frac{n}{\varepsilon^2\delta^2}\right)\right)$ bits

with probability at least $1 - \delta$

# Boosting via Chernoff

**Lem:** For $t \geq \boxed{8\log(2/\delta)}$ we have

$$\mathbb{P}[|\tilde{n}^{++} - n| \geq \varepsilon n] \leq \delta$$

Morris+: average over $s$ runs of Morris, then

$$\mathbb{P}[|\tilde{n}^{+} - n| \geq \varepsilon n] \leq \frac{1}{2s\varepsilon^2}$$

$\ell = t/4, \; n = t$

**Chernoff:** $\mathbb{P}[|X - \mathbb{E}[X]| \geq \ell] \leq 2\exp\left(-\frac{2\ell^2}{n}\right)$

for $X = X_1 + \cdots + X_n$, independent $X_1, \dots, X_n \in \{0,1\}$

**Proof:**

Each run of Morris+ succeeds with prob. $\geq \frac{3}{4}$

$$Y_i = \begin{cases} 1 & \text{if } i\text{-th run of Morris+ succeeds} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[Y_i] = \mathbb{P}[Y_i = 1] \geq \frac{3}{4}$

$Y = Y_1 + \cdots + Y_t$ $\qquad$ $\underline{\mathbb{E}[Y] \geq \frac{3}{4}t}$

$$\mathbb{P}[\text{Morris++ fails}] \leq \mathbb{P}\left[Y \leq \frac{t}{2}\right]$$

$$\leq \mathbb{P}\left[Y \leq \mathbb{E}[Y] - \frac{t}{4}\right] \leq 2\exp\left(-\frac{2t^2}{16\,t}\right) \leq \boxed{\delta}$$

**Morris++:** *median over t runs of Morris+* $\delta = \frac{1}{4}$

1) Run $t$ copies of Morris+ with $s := \lceil 2/\varepsilon^2 \rceil$
2) output **median** of their estimates $\tilde{n}_1^{+}, \dots, \tilde{n}_t^{+}$

(that is, sort and pick the middle value)

# Approximate Counting

Goal: $O(\log\log n)$ space

monitor a sequence of events, maintain an **approximate counter** of the number of events

**Data structure problem:**

maintain a number $n$

**update():** increment $n$ by 1

**query():** output $\tilde{n}$ with

$$\mathbb{P}[|\tilde{n} - n| \geq \varepsilon n] \leq \delta$$

**Solution:** Morris' Counter (1978)

1) Initialize $X = 0$
2) On update(): Increment $X$ with probability $2^{-X}$
3) On query(): output $\tilde{n} = 2^X - 1$

**Morris+:** *average over* $s = \lceil 2/\varepsilon^2 \rceil$ *runs of Morris*

**Morris++:** *median of* $t = \lceil 8\log(2/\delta) \rceil$ *runs of Morris+*

**Space usage:**

$O\left(\frac{1}{\varepsilon^2}\log\left(\frac{1}{\delta}\right)\log\log\left(\frac{n}{\varepsilon\delta}\right)\right)$ bits

with probability at least $1 - \delta$

# Approximate Counting

Goal: $O(\log \log n)$ space

monitor a sequence of events, maintain an **approximate counter** of the number of events

We learned about:

- probability basics
- concentration inequalities
- unbiased estimators
- boosting via Chebyshev
- boosting via Chernoff

→ *Primer to Randomness*

**Solution:** Morris' Counter (1978)

1) Initialize $X = 0$

2) On update(): Increment $X$ with probability $2^{-X}$

3) On query(): output $\tilde{n} = 2^X - 1$

**Morris+:** *average over* $s = \lceil 2/\varepsilon^2 \rceil$ *runs of Morris*

**Morris++:** *median of* $t = \lceil 8 \log(2/\delta) \rceil$ *runs of Morris+*

# More Material

- These slides will be available on the course website
- Video recording will be made available
- Course Website → Material → A Primer to Randomness
- Course Website → Material → Link to Summer School on Streaming by Jelani Nelson

- **Presence Exercise Sheet**: Will be send out in the next couple of days, Tutorial on May 11

**See you next week!**

# EXTRA