Geometric Algorithms with Limited Resources

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Assignments: biweekly, hand in 50% of total point value to take exam.
Exam: oral, soon after end of teaching.

Lectures are recorded (without video) and uploaded, see mailing list for access.

Geometric Algorithms with Limited Resources

Typical input: set of points or a metric space.
But! Not proper computatinal geometry
course, only what we need.

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Sublinear time, property testing.
Sublinear space, streaming.

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Geometric Algorithms with Limited Resources

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Sublinear space, streaming.

Recent (mostly after 2000) results, fresh research questions!

Introduction, concepts from computational geometry

Sándor Kisfaludi-Bak

Geometric algorithms with limited resources
Summer semester 2021

max planck institut
informatik

Overview

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- Computational models, limitations in space

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- Naive convex hull, gift wrapping in O(1) space

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- A classic deterministic algorithm in $\mathbb{R}^2$

Convex hull

Low-dim linear programming

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- A classic deterministic algorithm in $\mathbb{R}^2$
- Sublinear space LP (Chan–Chen '07)

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Real RAM vs. Word RAM

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arbitrary real numbers	words of size $\Theta(\log n)$

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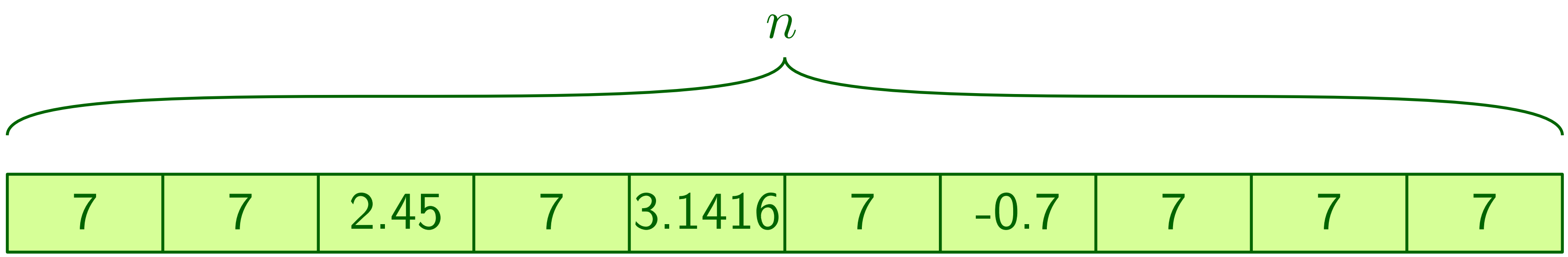
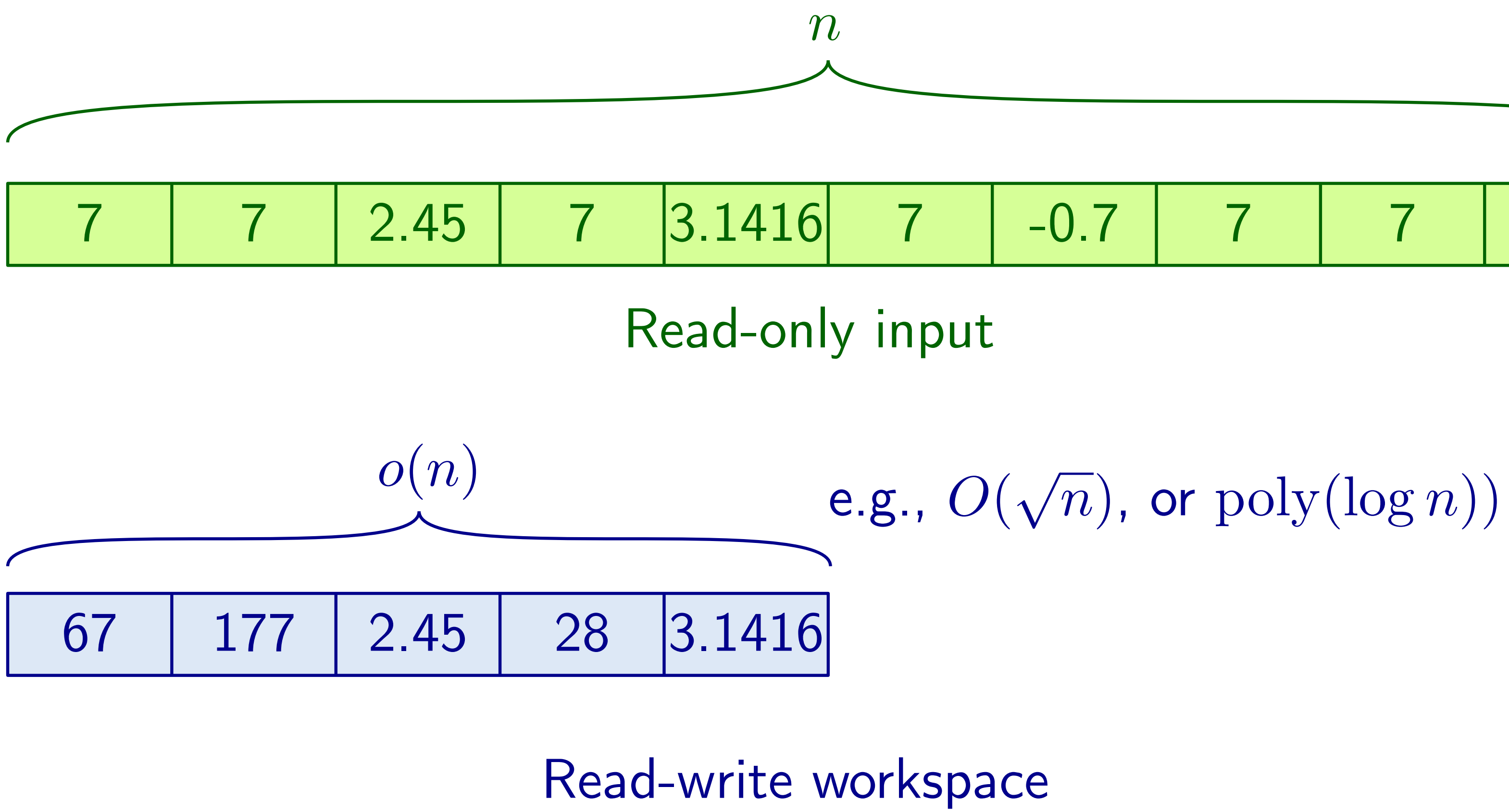
Unrealistic power	Too restrictive?
Often needed for exact computation	Usually enough for approximations!

Limited workspace model



Read-only input

Limited workspace model



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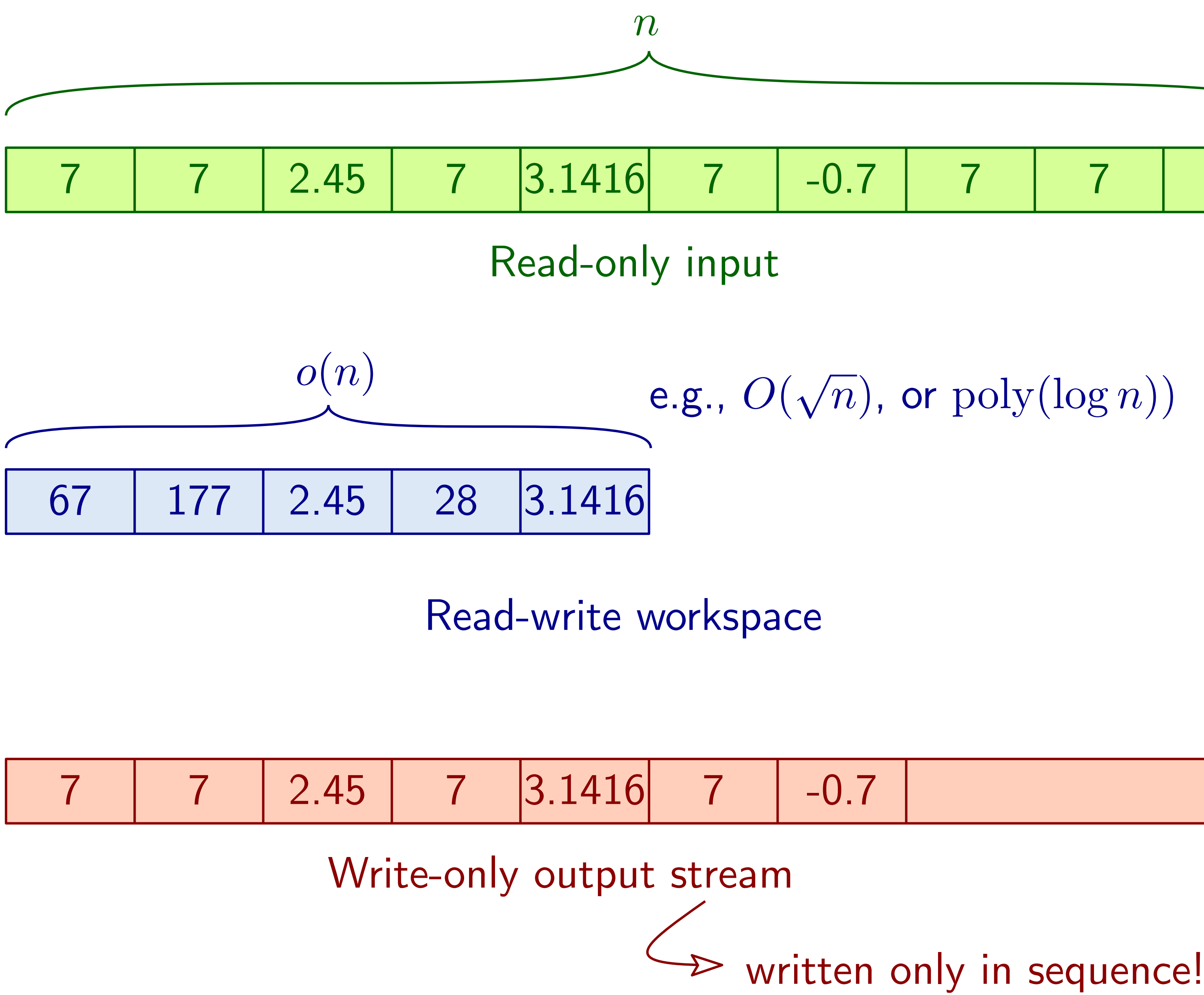
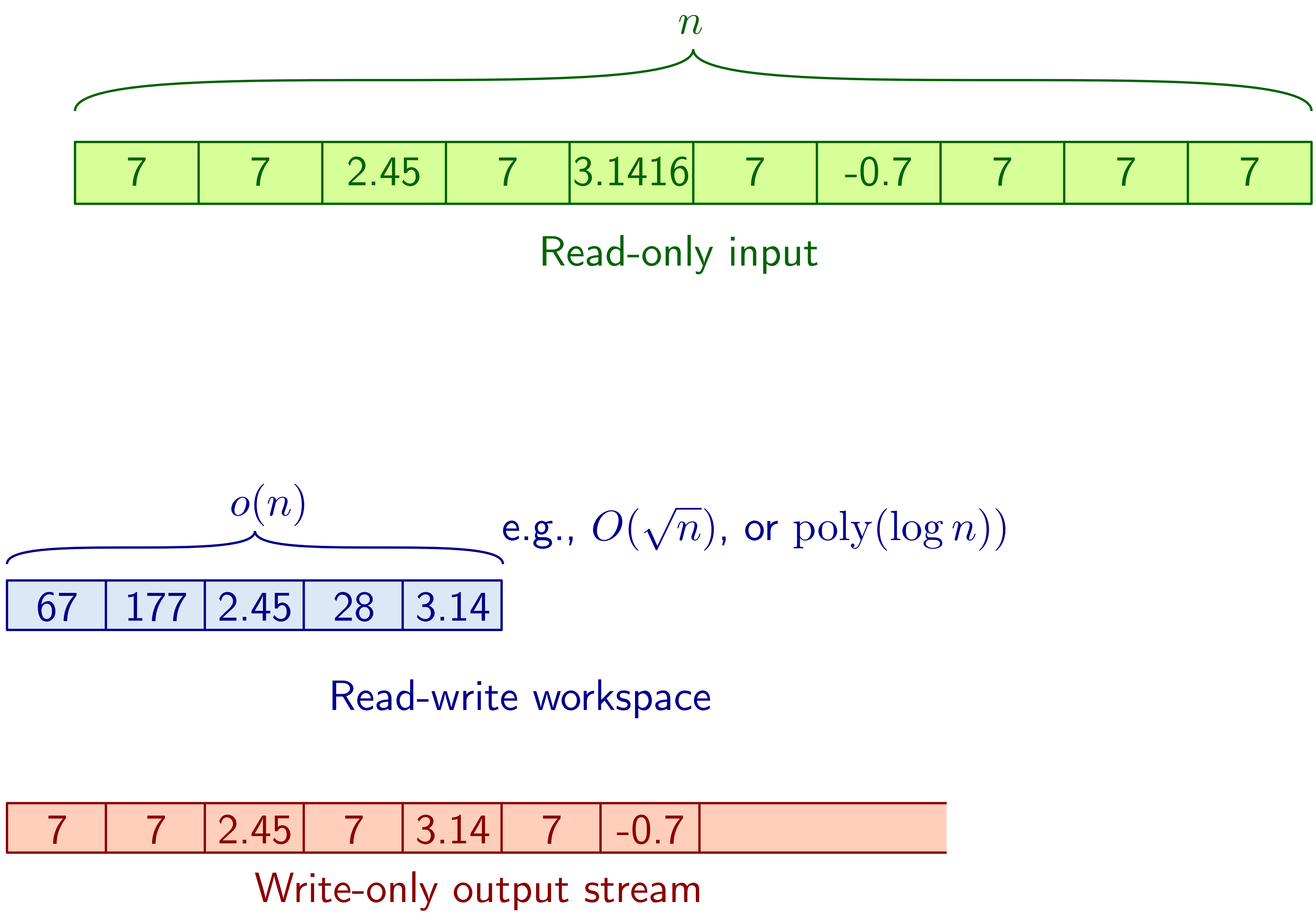


Streaming and multi-pass model



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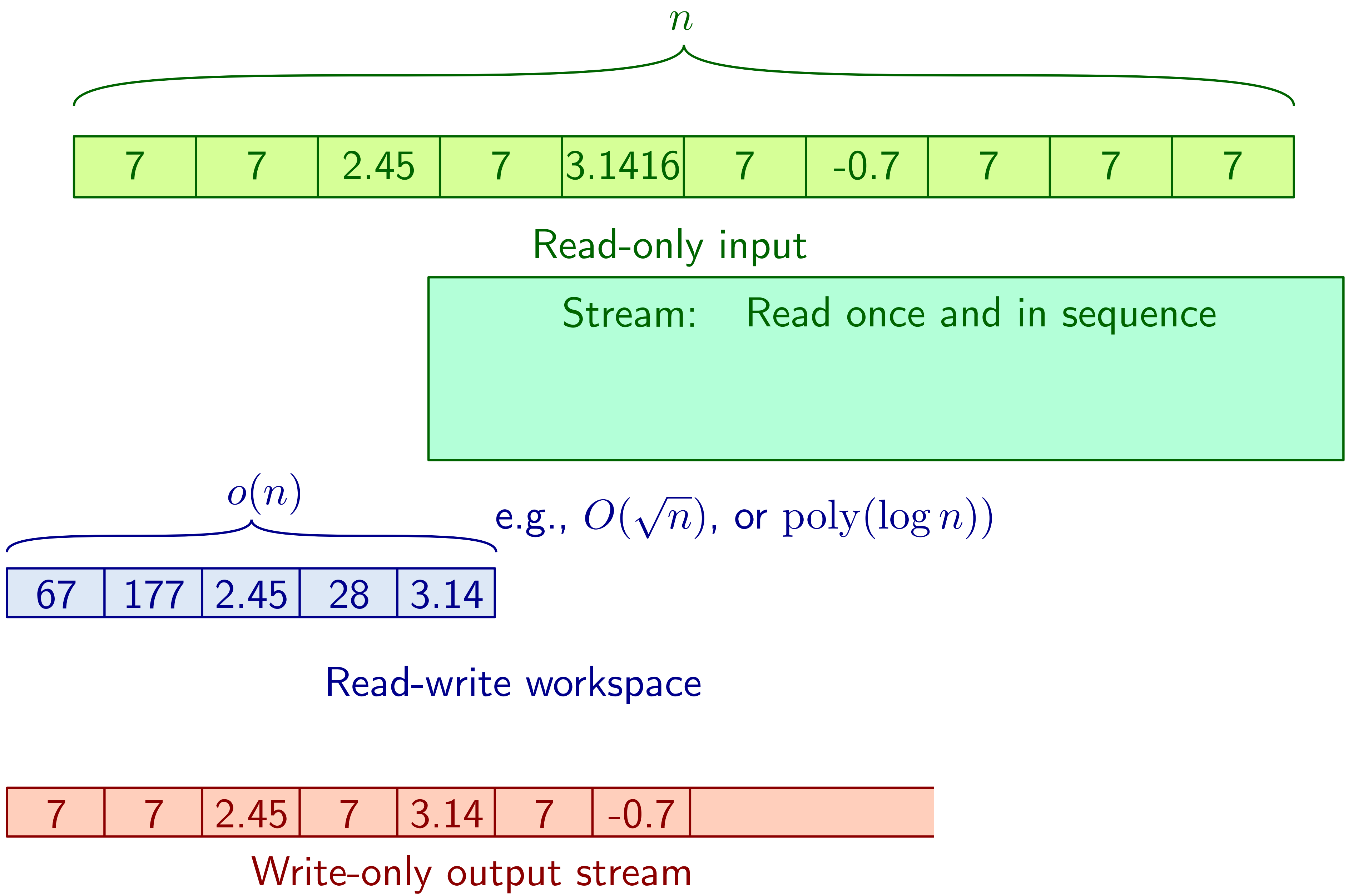

Streaming and multi-pass model

n

7	7	2.45	7	3.1416	7	-0.7	7	7	7

Read-only input

Stream:	Read once and in sequence
multi-pass, k-pass:	Read k times and in sequence

$o(n)$ e.g., $O(\sqrt{n})$, or $\mathrm{poly}(\log n))$

67	177	2.45	28	3.14

Read-write workspace

7	7	2.45	7	3.14	7	-0.7	

Write-only output stream

Convex hull

Notations, definitions

$\mathbb{R}^d$ is d-dimensional Euclidean space

$P = \{p_1, \ldots, p_n\}$ set of n points

$X \subseteq \mathbb{R}^d$ is *convex* if for any $p, q \in X$ we have $pq \subseteq X$

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Convex hull:

$$\mathrm{conv}(P) = \begin{cases} \text{minimum convex set containing } P \\ \text{intersection of convex sets containing } P \\ \{\alpha_1 p_1 + \cdots + \alpha_n p_n \mid \alpha_i \geq 0 \text{ and } \sum_{i=1}^n \alpha_i = 1\} \end{cases}$$

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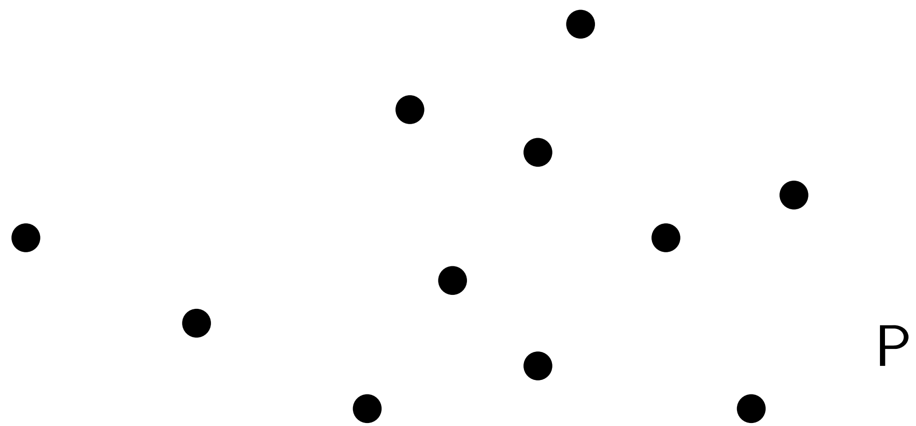

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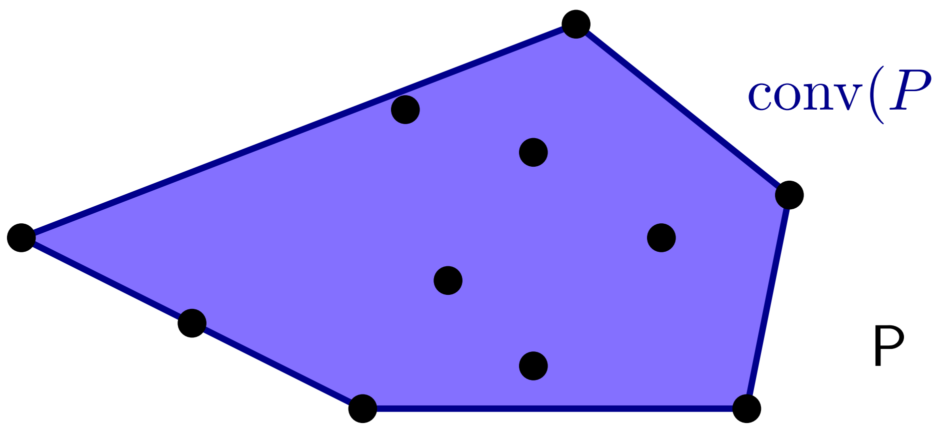
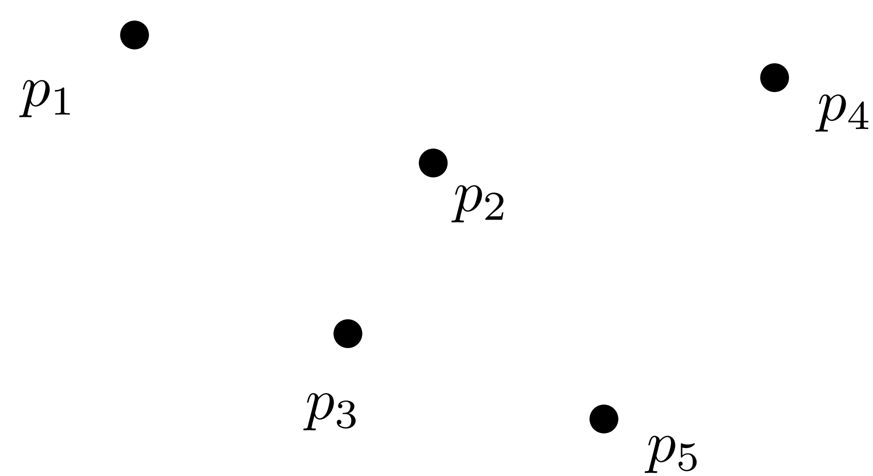


Convex hull: input and output

Input: Points with coordianate pairs $(x, y) \in \mathbb{R}^2$

$$(e, \pi), (3, 3), (2.95, 2.9), (\sqrt{11}, 3.05), (\pi, e)$$



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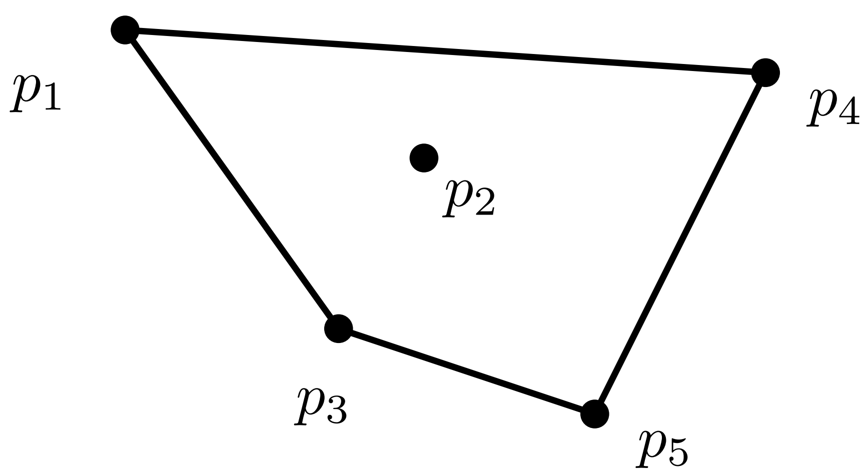
Output: "corners" in clockwise order
smallest $Q \subseteq P$ s.t. $\mathrm{conv}(Q) = \mathrm{conv}(P)$

$$p_1, p_4, p_5, p_3$$



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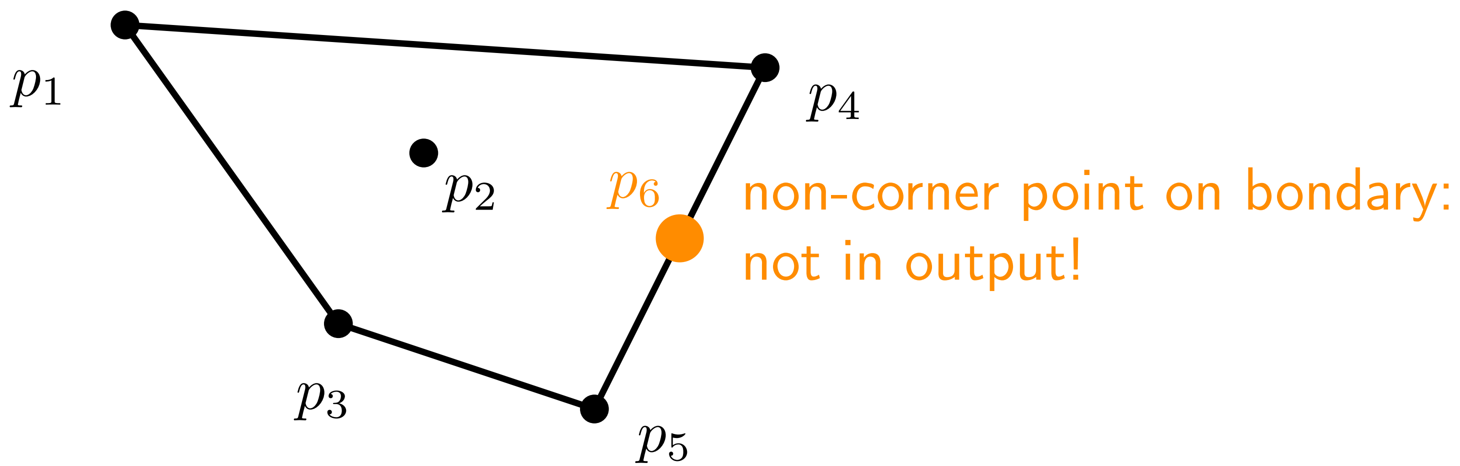

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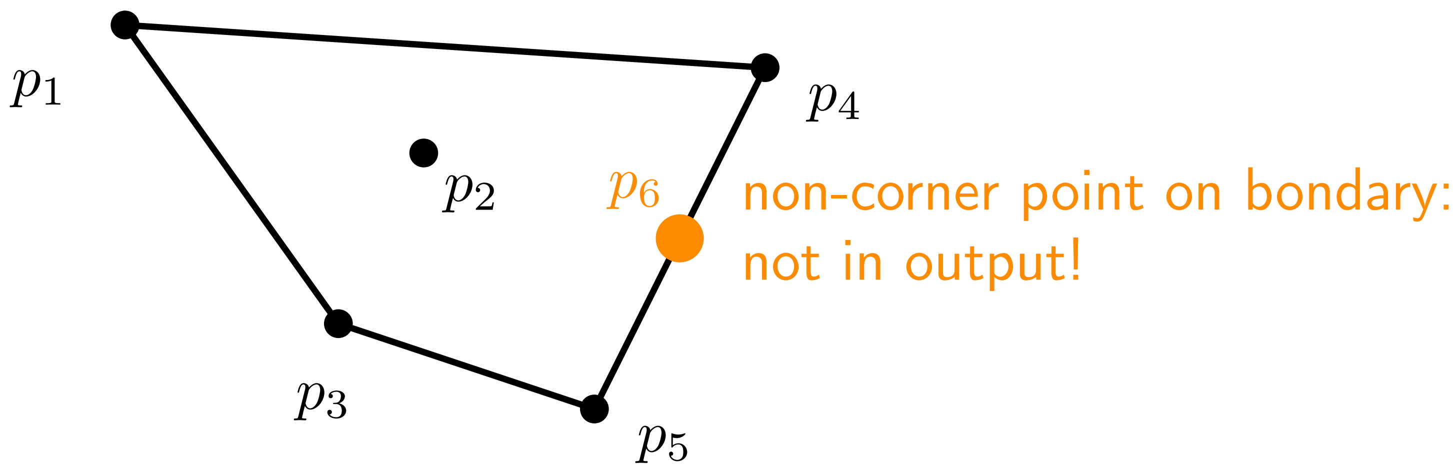

Everything works with rational inputs on Word RAM!

Naive algorithms, workspace O(1)

Naive Algorithm

Suppose no 3 points on one line. (no *collinear triples*)

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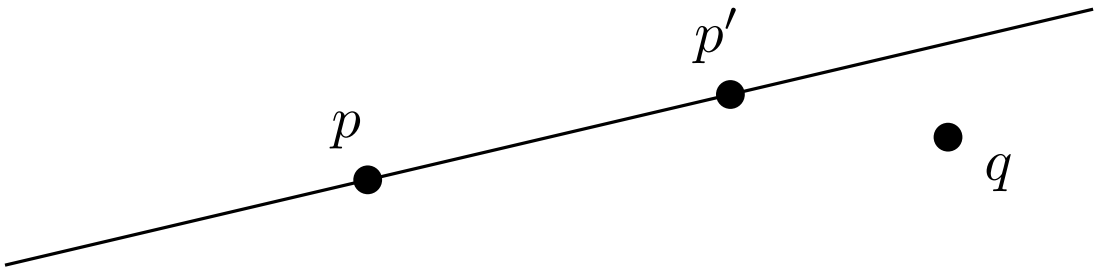
In $O(1)$ time, decide if q is on left or right side of line pp'



$$\operatorname{sgn}\left(\begin{vmatrix} p_x & p_y & 1 \\ p'_x & p'_y & 1 \\ q_x & q_y & 1 \end{vmatrix}\right)$$

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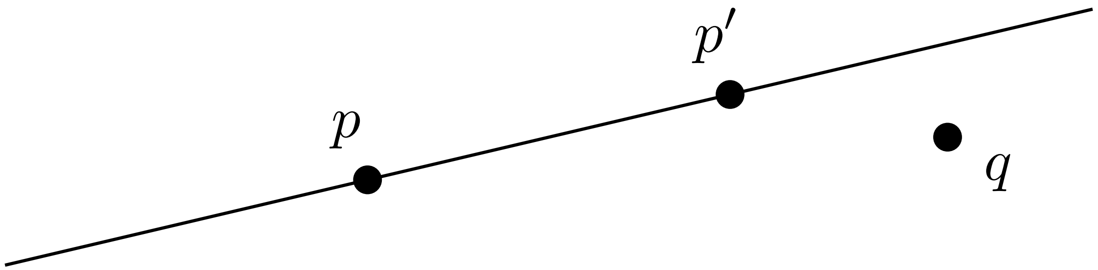

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Naive Convex Hull in $\mathbb{R}^2$

For each $p, p' \in P$,

check if all $q \in P \setminus \{p, p'\}$ is on the left of line pp'.

If yes, then p' follows p in $\operatorname{conv}(P)$.

Assemble and output the hull

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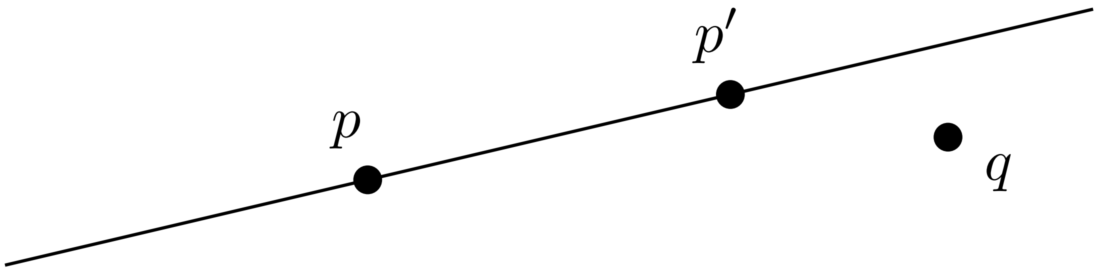

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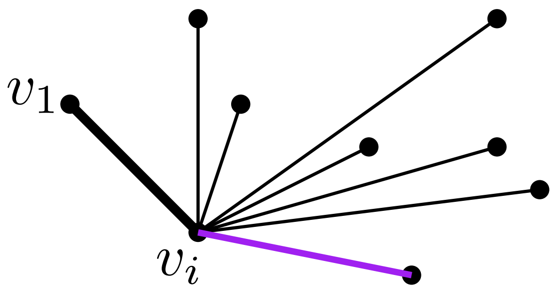
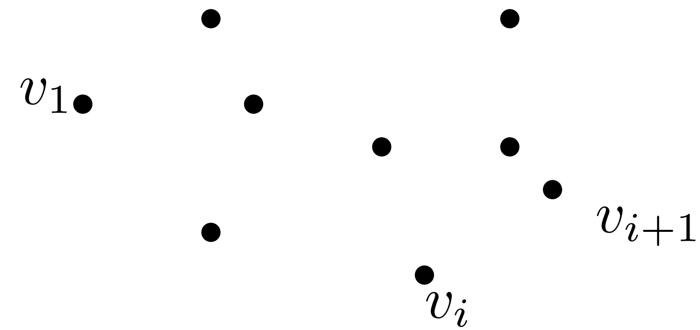
Running time: $\binom{n}{2} \cdot (n-2) \cdot O(1) = O(n^3)$

Less naive algorithm: Jarvis' March – aka gift wrapping

Algorithm:
Start at leftmost point, find next point with minimum (maxmum) slope.





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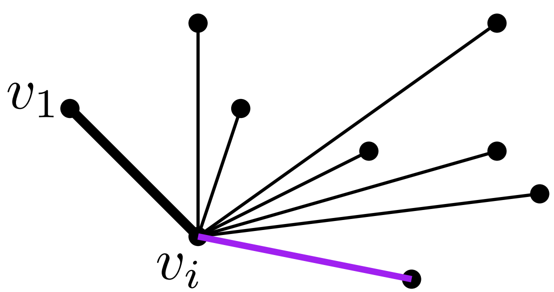

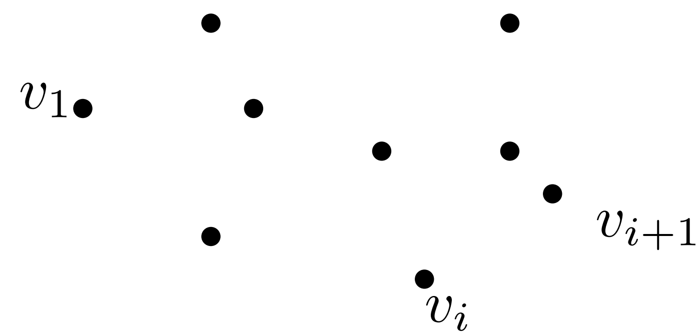

$O(hn)$ time, and enough to keep track of v_1, v_i, v_{i+1}. $O(1)$ space.

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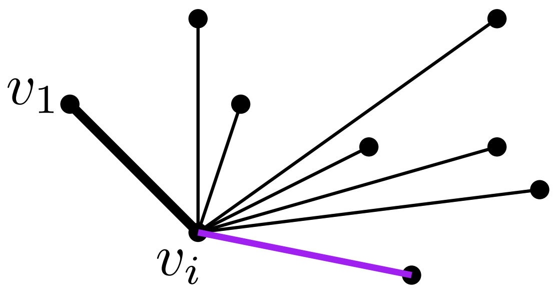

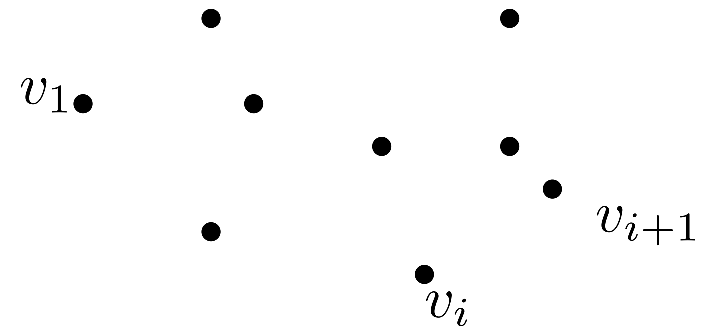

$O(hn)$ time, and enough to keep track of v_1, v_i, v_{i+1}. $O(1)$ space.

$h =$ size of convex hull. Output-sensitive algorithm.

Graham's scan (1972)

Graham's Scan idea

Suppose points have distinct x-coordinates.

Let $p_1, \ldots, p_n$: points sorted with increasing x-coordinates.

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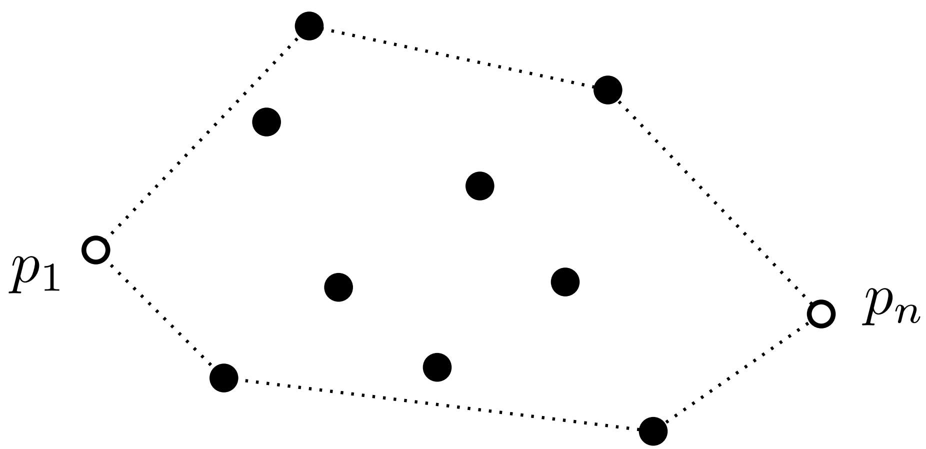
$\longrightarrow$ p_1, p_n are on convex hull

Upper hull
part of the hull after p_1 and before p_n in clockwise order



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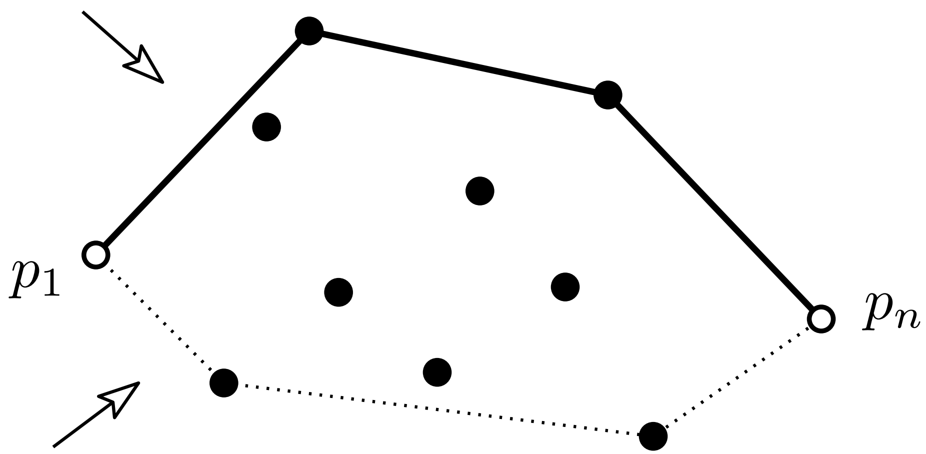


Lower hull

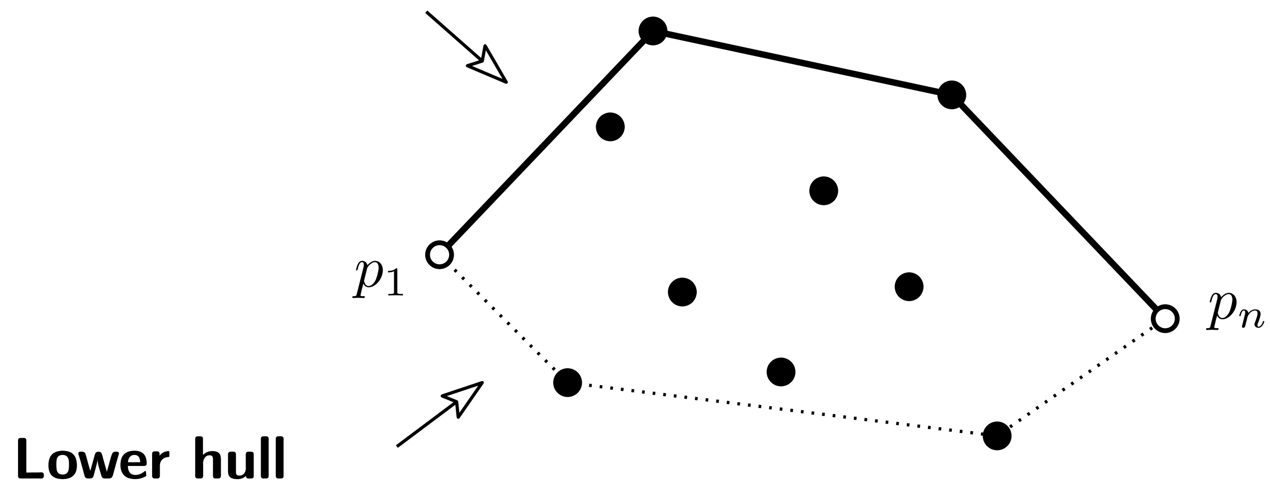
Graham's Scan idea

Suppose points have distinct x-coordinates.

Let $p_1, \ldots, p_n$: points sorted with increasing x-coordinates.

$\rightarrow$ p_1, p_n are on convex hull

Upper hull
part of the hull after p_1 and before p_n in clockwise order

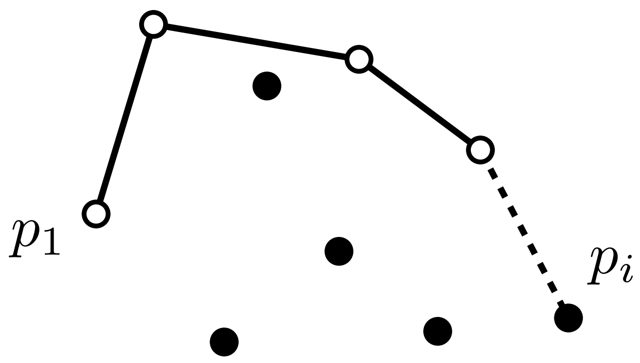


Idea:
Add points left to right, update upper hull after each addition

Graham's Scan: update

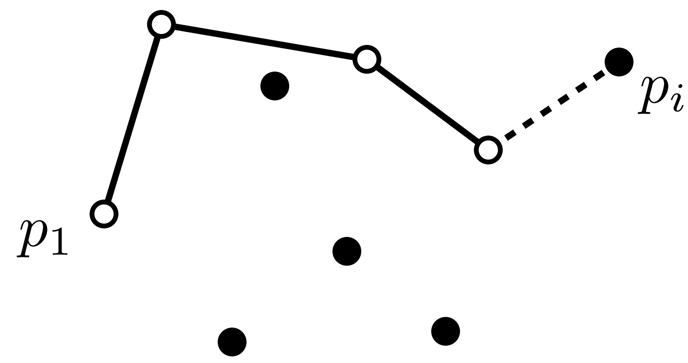
Right turn

(p_i is below last hull segment)



Left turn

(p_i is above last hull segment)



Graham's Scan: update

Right turn

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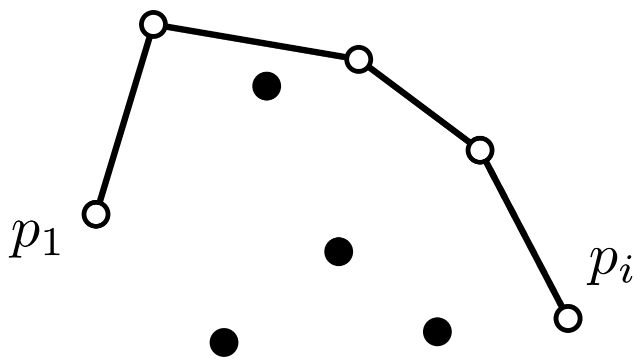
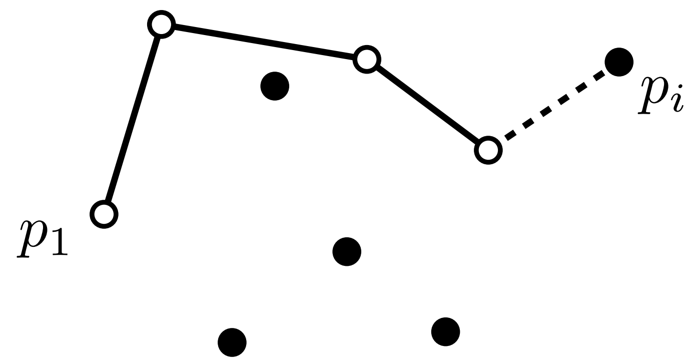


Add p_i to the upper hull

Left turn

(p_i is above last hull segment)



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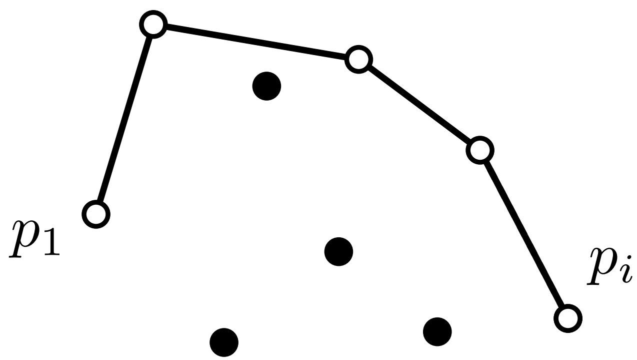

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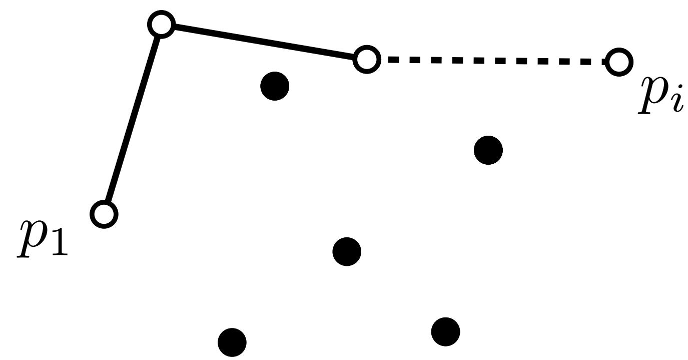

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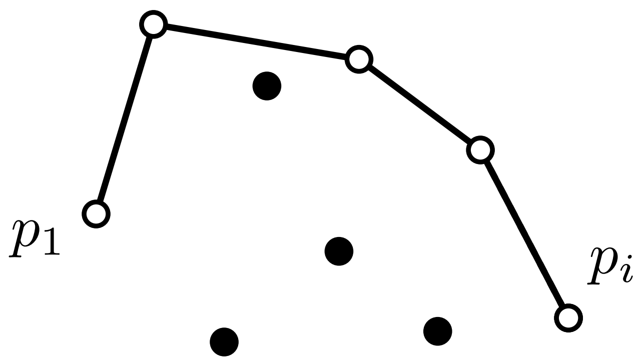

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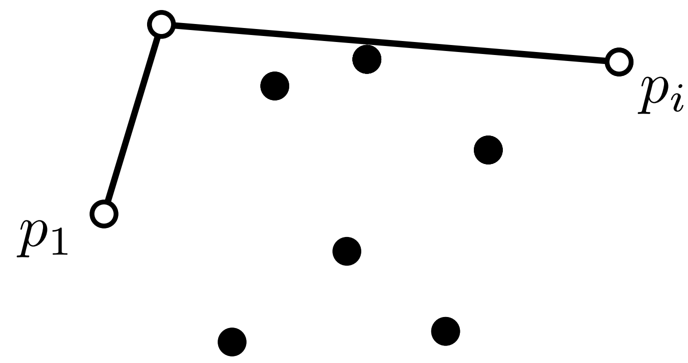
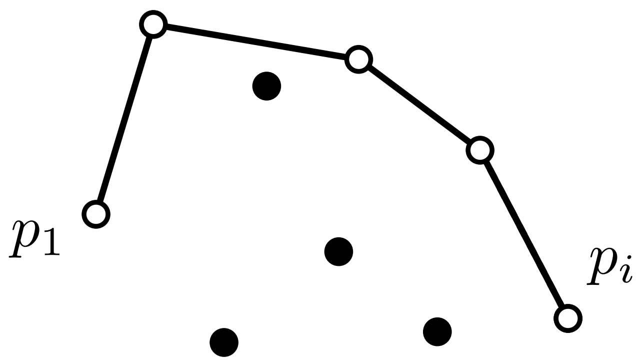


Add p_i but remove previous hull point until left turn disappears

Graham's Scan: update

Right turn

(p_i is below last hull segment)



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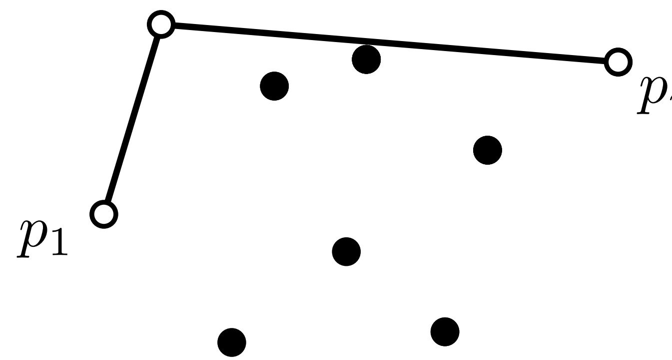

Add p_i but remove previous hull point until left turn disappears

Similalrly for lower hull, after adding p_i:
while last three points of lower hull q, q', p_i are a right turn:
remove the middle point q'

Graham's Scan: pseudocode + runtime

```
Sort P by increasing x-coordinates
Add p_1, p_2 to U and L
for i = 3 to n do
    Add p_i to U and L
    while last three pts of U form left turn do
        Remove pt preceding p_i from U
    while last three pts of L form right turn do
        Remove pt preceding p_i from L
return L and reverse of U
```

Graham's Scan: pseudocode + runtime

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Running time:

Sorting ⟶ $O(n \log n)$

Graham's Scan: pseudocode + runtime

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Sort P by increasing x-coordinates
Add p1, p2 to U and L
for i = 3 to n do
    Add pi to U and L
    while last three pts of U form left turn do
        Remove pt preceding pi from U
    while last three pts of L form right turn do
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return L and reverse of U
```

Running time:

Sorting $\longrightarrow O(n \log n)$

Each $p \in P$ is:

- added once to U (same for L) $\longrightarrow O(n)$
- removed at most once from U (same for L) $\longrightarrow O(n)$

Triplets checked in While loop heads $\longrightarrow O(n)$

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$O(n \log n)$ time, but $O(n)$ space.

Time-optimal because of sorting (was exercise last year)

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Near-optimal time-space tradeoff:
sorting on RAM requires $T \cdot S = \Omega(n^2 / \log n)$. [Borodin–Cook '82]

Convex hull with good time-space tradeoff

Sorting in sublinear space

Theorem (Munro, Paterson 1980)
Given x and an unsorted array A, we can find the s smallest elements greater than x in A in a single pass, in $O(s)$ space and $O(n)$ time.
We can also sort in

- $O(n^2/s + n \log s)$ time
- $O(s)$ space
- with n/s passes.

Convex hull in sublinear space

Theorem (Chan–Chen 2007)
Given n points in $\mathbb{R}^2$, the convex hull can be computed in

- $O(n^2/s + n \log s)$ time
- $O(s)$ space
- with n/s passes.

Sublinear space convex hull pseudocode

$v :=$ leftmost point
while $v \neq$ rightmost point **do**
    Find vertical slab σ with s pts whose left wall contains v
    $q_0, \ldots, q_j =$ upper hull of $P \cap \sigma$
    for all $p \in P$ to the right of σ **do**
        while $q_{j-1} q_j p$ is left turn **do**
            $j := j - 1$
        $j := j + 1, \quad q_j := p$
    Print$(q_0, \ldots, q_j)$
    $v := q_j$

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$2\lceil n/s \rceil$ passes, $O(s)$ space, $O((n/s) \cdot (n + s \log s))$ time

Linear Programming in low-dimensional space

LP with 2 variables: halfplanes in $\mathbb{R}^2$

Given:

$$\min c_1 x + c_2 y \text{ subject to}$$
$$a_{11}x + a_{12}y \leq b_1$$
$$a_{21}x + a_{22}y \leq b_2$$
$$\cdots$$
$$a_{n1}x + a_{n2}y \leq b_n$$

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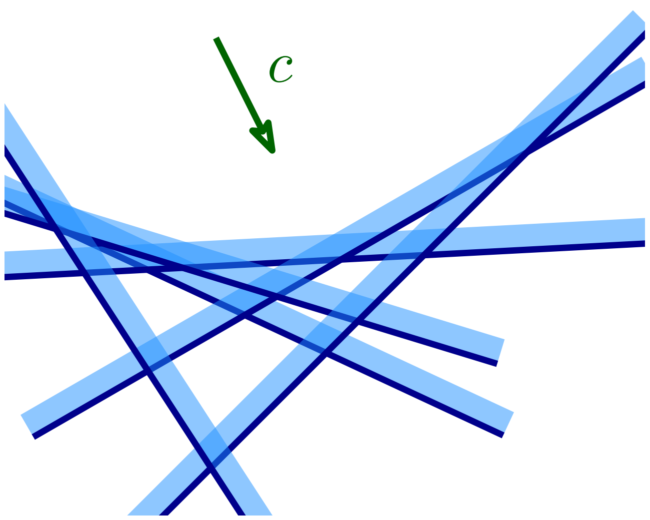
Given set H of n halfplanes, find extreme point in direction c.



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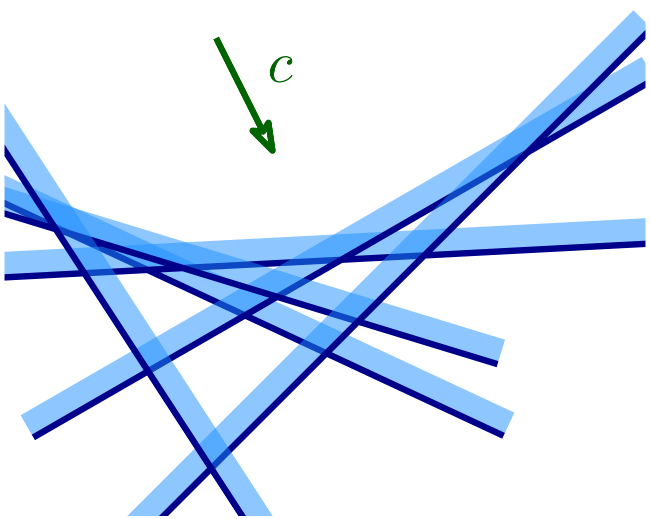

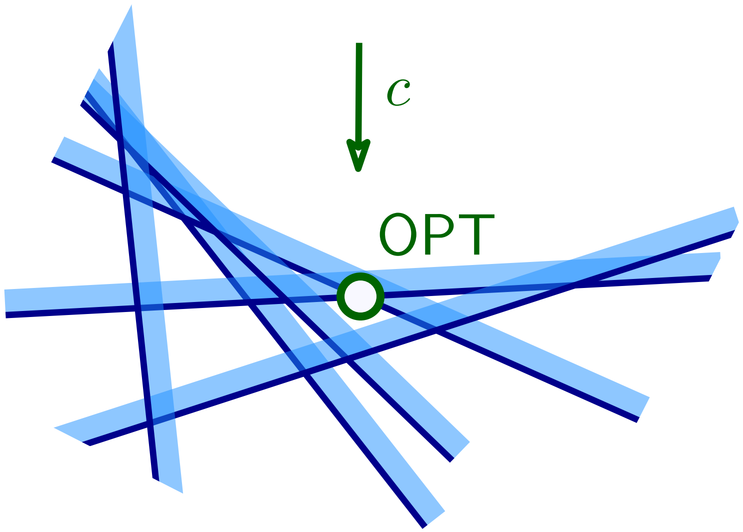

Dual Graham's scan solves it in $O(n \log n)$

Deterministic method: paired halfplanes

Lemma [Megiddo, Dyer 1984]
Assuming that $\bigcap_{h \in H} H \neq \emptyset$ is bounded from below, we can find OPT in $O(n)$ time.

Sublinear space low-dimensional LP

We prove:

Theorem (Chan–Chen 2007)
Fix $\delta > 0$. Given n half-planes in $\mathbb{R}^2$, the lowest point of their intersection can be computed in

- $O(\frac{1}{\delta}n^{1+\delta})$ time
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We might return to:

Theorem (Chan–Chen 2007)
Given n half-spaces in $\mathbb{R}^d$ and $\delta > 0$, the lowest point of their intersection can be computed in

- $O_d(\frac{1}{\delta^{O(1)}} n)$ time
- $O_d(\frac{1}{\delta^{O(1)}} n^{\delta})$ space
- with $O(1/\delta^{d-1})$ passes.

Theorem (Chan–Chen 2007)
Given n half-spaces in $\mathbb{R}^d$, the lowest point of their intersection can be computed in $O_d(n)$ time and $O_d(\log n)$ space.

Towards sublinear space LP: filtering and listing

Given stream H of halfplanes, produce stream of vertical lines.

List(r, σ, H)

while H not read through **do**

$h_1, \ldots, h_r :=$ next r halfplanes from H

Compute $I = h_1 \cap \cdots \cap h_r$

Print vertical lines through vertices of I that fall in σ

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Compute $I = h_1 \cap \cdots \cap h_r$
Print halfplanes involved in $\partial(I \cap \sigma)$

List and Filter work in one pass, in $O(r)$ space and $O(n \log r)$ time.

Sublinear time LP in $\mathbb{R}^2$

Parameter: r

Invariant: solution is in σ_i and defined by halfplanes in H_i

Pseudocode

LP(r, σ, H)
  $\sigma_0 := \mathbb{R}^2$
  for $i = 0, 1, \ldots$ **do**
    if $|H_i| = O(1)$ **then**
      return brute force solution for H_i
    Divide σ_i into r slabs with roughly same # of lines from $List_{r,\sigma_i}(H_i)$
    Decide which subslab has the solution, let that be σ_{i+1}.
    $H_{i+1} := Filter_{r,\sigma_{i+1}}(H_i)$

Preserves invariant ✓

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Pseudocode

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```
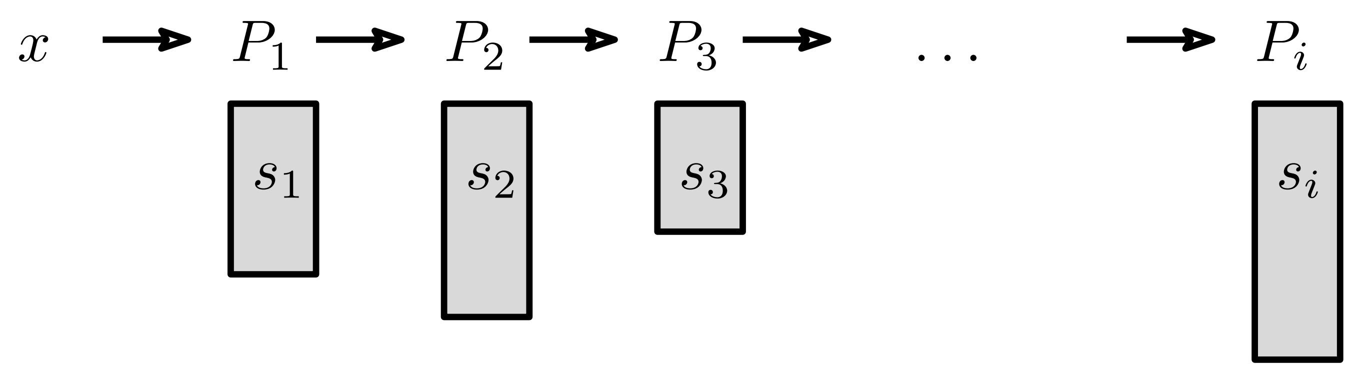

One pass, maintain $r-1$ minima at inner slab walls

Space-efficient pipeline of streams

How to execute $P_i(P_{i-1}(\ldots(P_1(x))))$
if P_j are single-pass processes with worksapce s_j and time t_j?

Pipeline:



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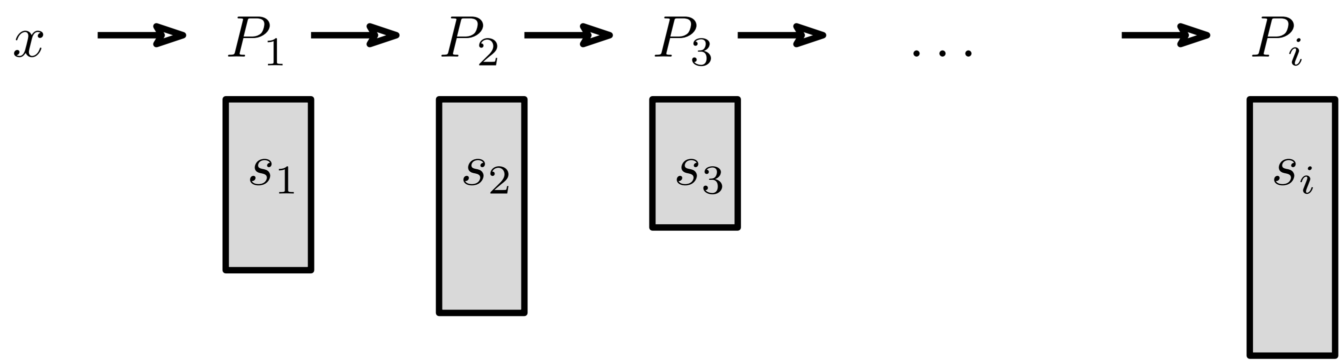

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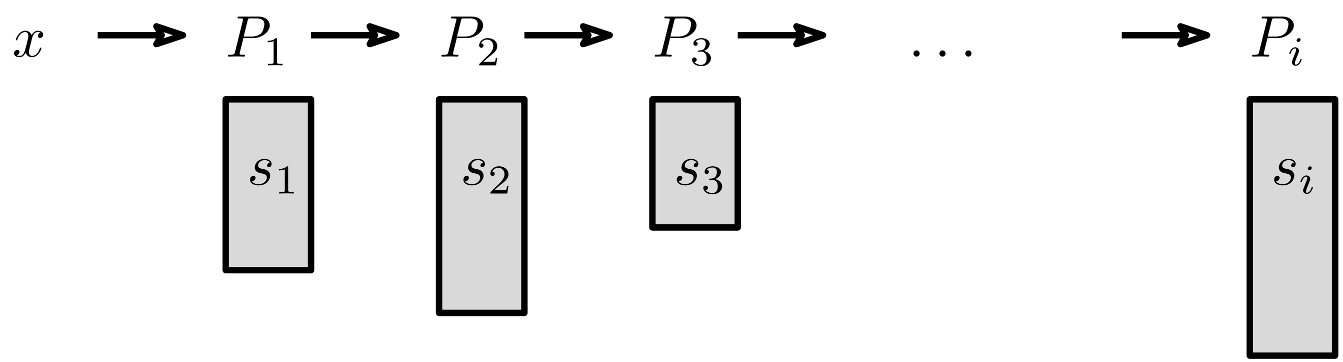

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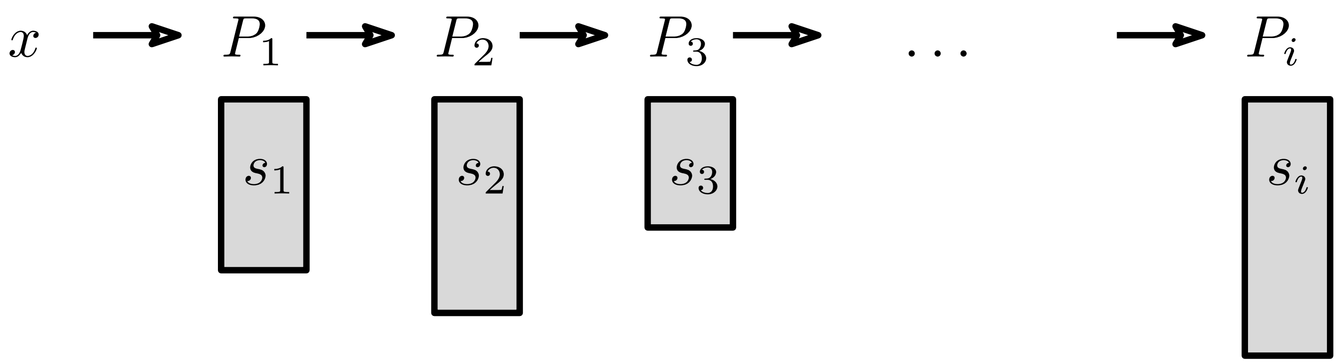

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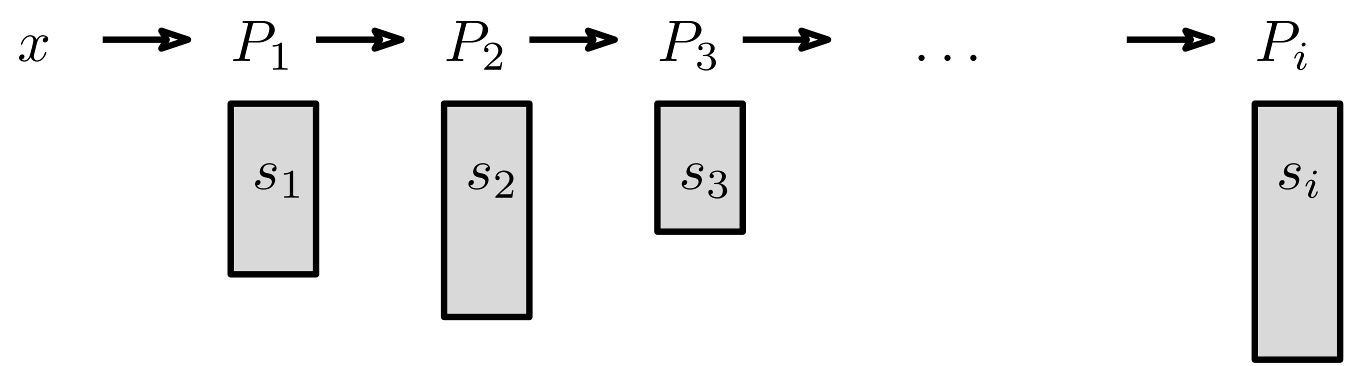

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Time and space needs

Filter(r, σ, H)

$\sigma_0 := \mathbb{R}^2$

for $i = 0, 1, \ldots$ **do**

if $|Filter_{r,\sigma_i}(\ldots(Filter_{r,\sigma_1}(H)))| = O(1)$ **then**

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Divide σ_i into r slabs:

$\mathbf{ApproxQuant_r}\big(List_{r,\sigma_i}(Filter_{r,\sigma_i}(\ldots(Filter_{r,\sigma_1}(H))))\big)$

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$O(ri) = O(r \log_r n)$ space, $O(n_0 \log r + \cdots + n_{i-1} \log r) = O(n \log r)$ time

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Altogether: $O(r \log_r n + r \log^2 n)$ space and $O(nr \log_r n)$ time.

Chan-Chen simple LP wrap-up

Theorem (Chan–Chen 2007)
Fix $\delta > 0$. Given n half-planes in $\mathbb{R}^2$, the lowest point of their intersection can be computed in

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Altogether: $O(r \log_r n + r \log^2 n)$ space and $O(nr \log_r n)$ time.

Set $r = n^{\delta/2}$.

$$O\left(n^{\delta/2} \cdot \tfrac{2}{\delta} + n^{\delta/2} \log^2 n\right) = O(\tfrac{1}{\delta} n^{\delta}) \qquad O\left(n \cdot n^{\delta/2} \cdot \frac{\log n}{\log n^{\delta/2}}\right) = O(\tfrac{1}{\delta} n^{1+\delta})$$

Approximate quantiles