Sublinear Intersection Testing

Sándor Kisfaludi-Bak

Geometric algorithms with limited resources
Summer semester 2021

max planck institut
informatik

Overview

- Searching in a jumbled sorted list optimally
- Intersection of convex polygons
- Testing if convex polytopes intersect with preprocessing
- Testing if convex polytopes intersect without preprocessing

Searching in an unsorted list

Given:
Doubly linked sorted list L
Stored in a size-n array

Theorem
There is an algrithm that finds the successor of x in L in $O(\sqrt{n})$ expected time.

Yao's principle

Theorem (Yao's principle, '77)
The expected cost of a randomized algorithm on the worst-case input is at least the expected cost of the best deterministic algorithm over the worst-case distribution of inputs.

Yao's principle

Theorem (Yao's principle, '77)
The expected cost of a randomized algorithm on the worst-case input is at least the expected cost of the best deterministic algorithm over the worst-case distribution of inputs.

$\mathcal{X}$: set of inputs, X: random input according to distribution q over $\mathcal{X}$.
$\mathcal{A}$: set of algorithms, A: random algorithm from distribution p over $\mathcal{A}$.

$$\max_{x \in \mathcal{X}} \mathbf{E}_p(cost(A, x)) \geq \min_{a \in \mathcal{A}} \mathbf{E}_q(cost(a, X))$$

Lower bound for searching an unsorted list

Theorem

There is no $o(\sqrt{n})$ expected time algorithm for successor finding in unsorted lists.

Convex polygon intersection

Theorem (Chazelle, Liu, Magen '06)
Given two convex polygons with cyclic list of vertices, we can decide if they intersect in $O(\sqrt{n})$ expected time.

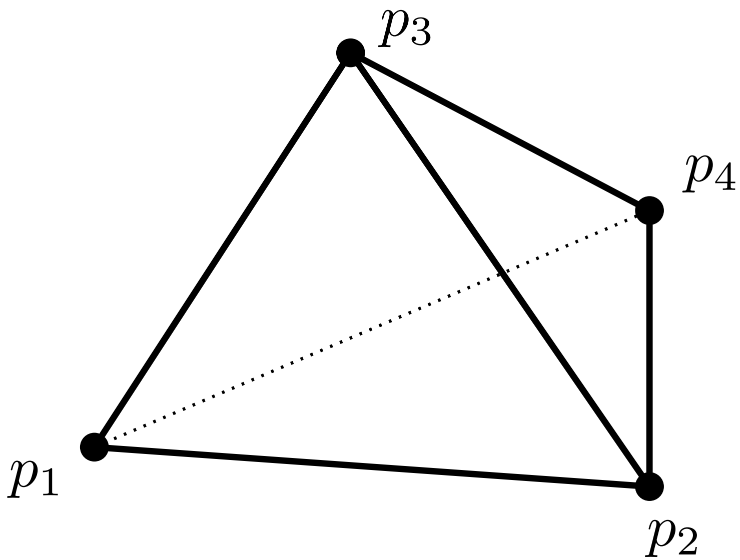
Convex polygon intersection

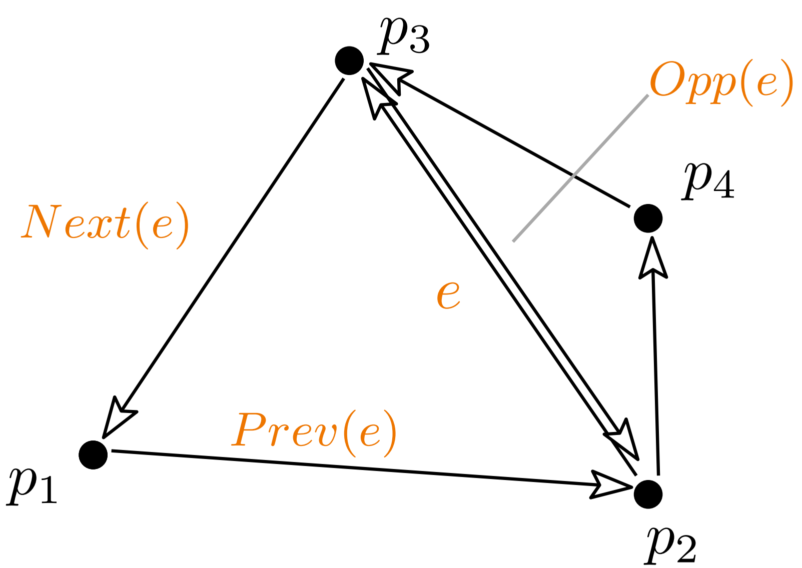
Theorem (Chazelle, Liu, Magen '06)
Given two convex polygons with cyclic list of vertices, we can decide if they intersect in $O(\sqrt{n})$ expected time.

P, Q polygons of size n, R_p, R_q samples of size r.

Lower bound for convex polygons

Convex polytope in $\mathbb{R}^3$ via DCEL





DCEL = Doubly connected edge list,
facets are ccw cycles from outside
arcs know: opposite, next, prev arc

Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

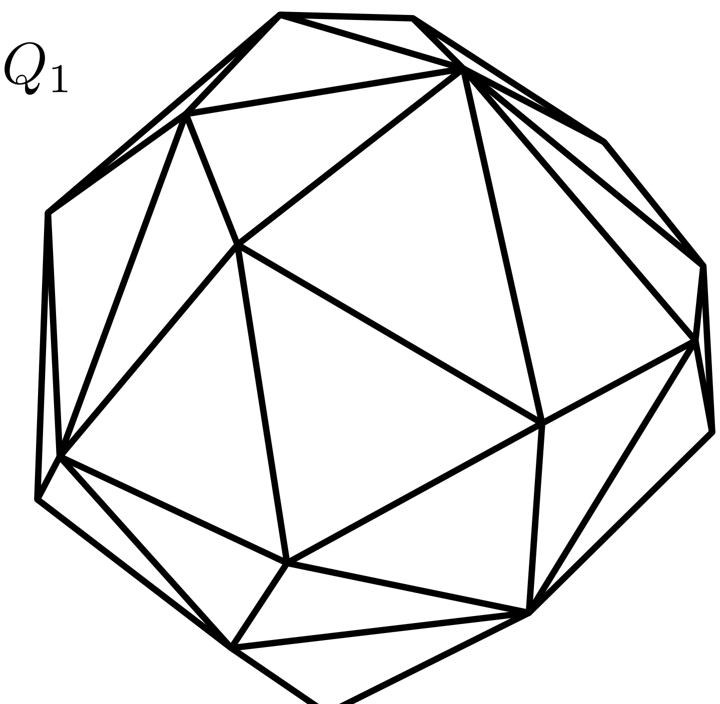
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.

Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.

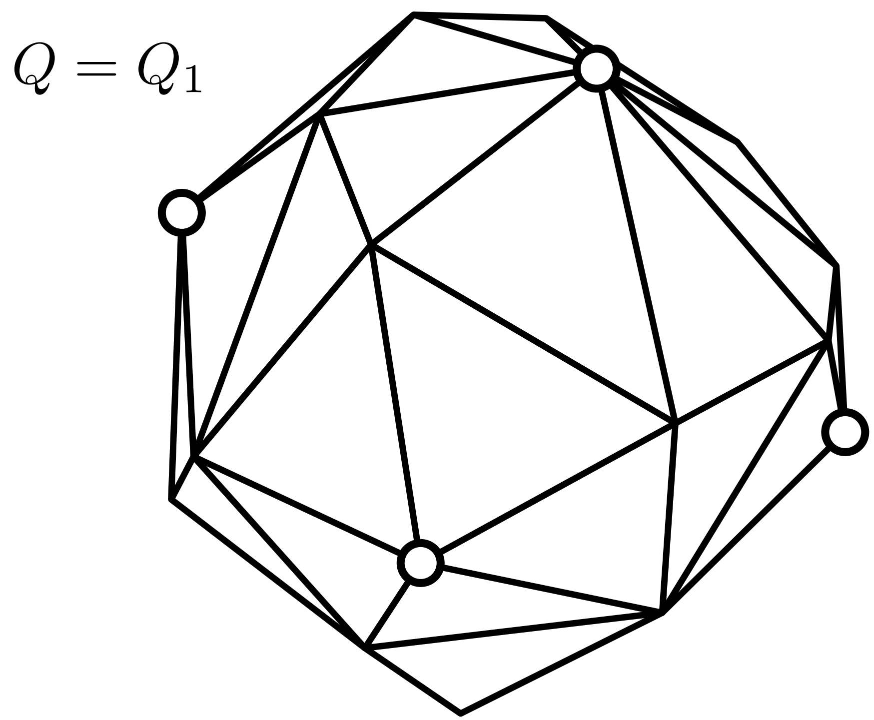
$Q = Q_1$



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

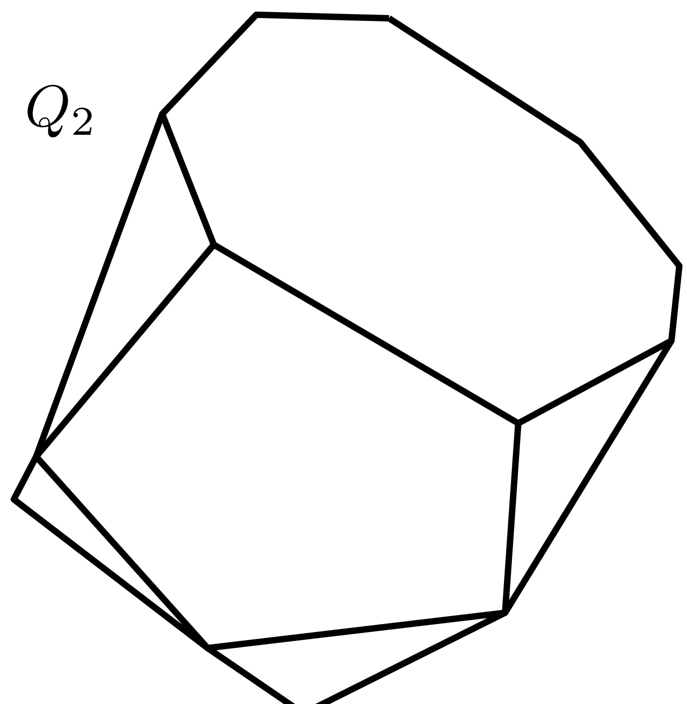
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

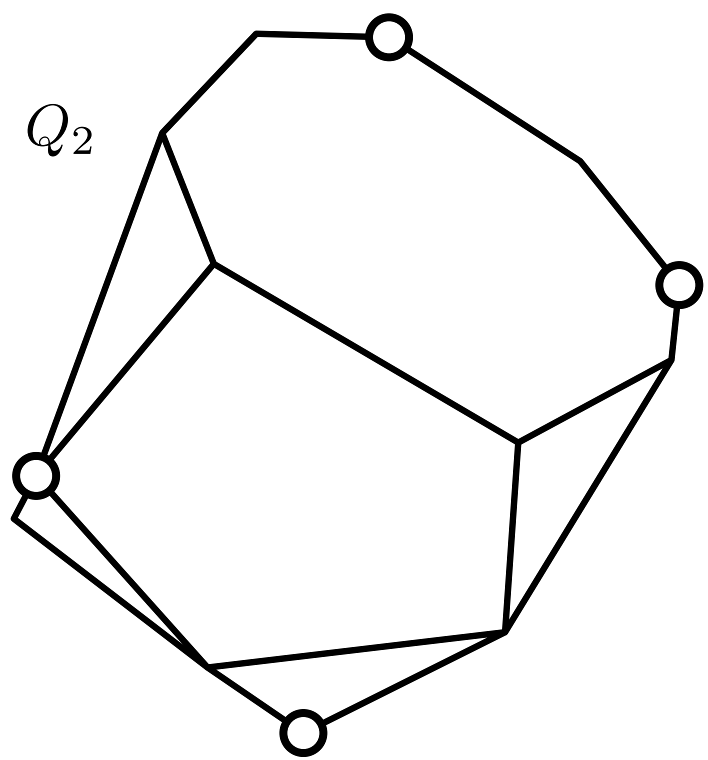
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

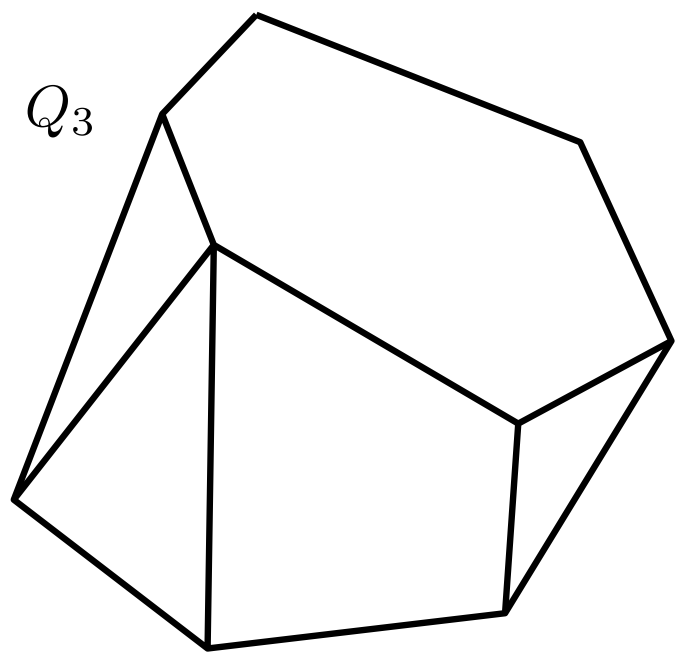
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

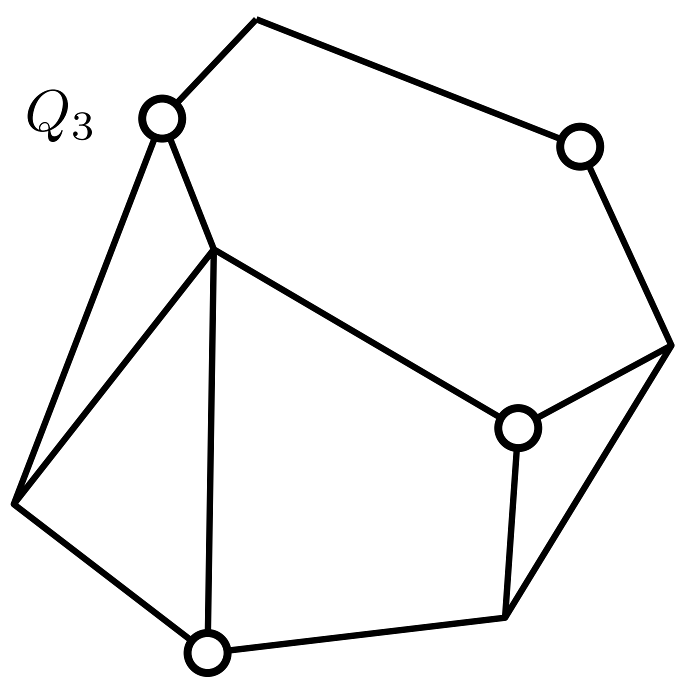
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

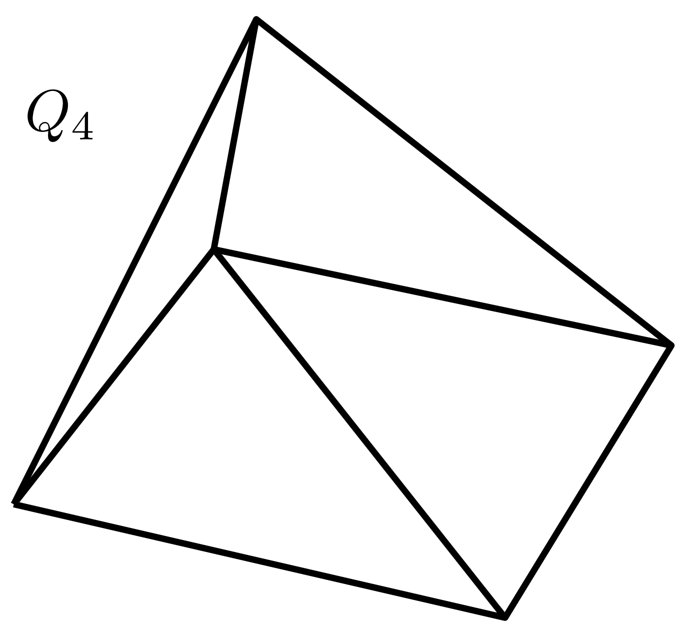
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

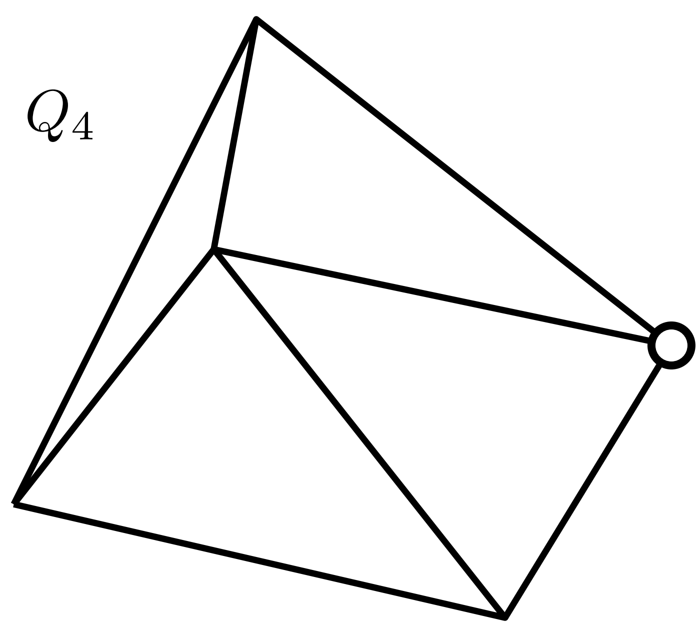
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

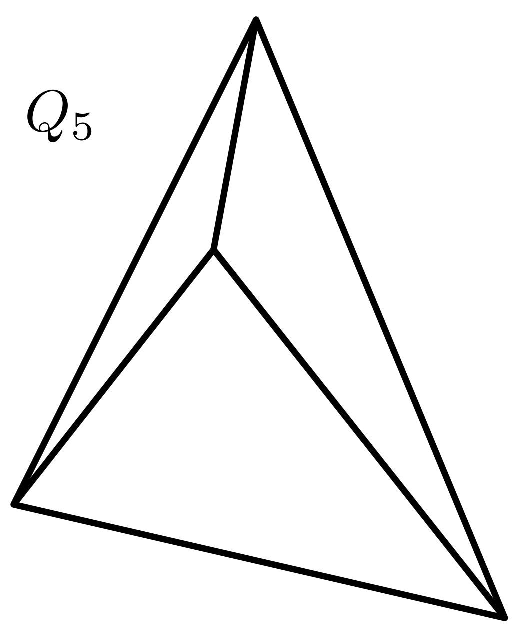
1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.



Constructing the DK hierarchy

Theorem Given Q via DCEL, a DK hierarchy of
- depth $k = O(\log n)$,
- size $\sum_{i=1}^{k}(|V(Q_i)|) = O(n)$,
- and degree $\max_i \max\{\deg_{G(Q_i)}(v) \mid v \in V(Q_i) \setminus V(Q_{i+1})\} \leq 11$

can be computed in $O(n)$ time.

Constructing the DK hierarchy

Theorem Given Q via DCEL, a DK hierarchy of
- depth $k = O(\log n)$,
- size $\sum_{i=1}^{k}(|V(Q_i)|) = O(n)$,
- and degree $\max_i \max\{\deg_{G(Q_i)}(v) \mid v \in V(Q_i) \setminus V(Q_{i+1})\} \leq 11$

can be computed in $O(n)$ time.

Proof. Iteratively remove set S, a greedy maximal independent set among vertices of degree ≤ 11.

Claim: $|S| \geq |V(Q)|/24$.
Suppose not: $|S| < |V(Q)|/24$
$\Rightarrow \bigcup_{s \in S} N[s] < |V(Q)|/2$
$\Rightarrow G(Q)$ has $\geq |V(Q)|/2$ vertices of degree ≥ 12
$\Rightarrow G(Q)$ has $\geq (|V(Q)|/2) \cdot 12/2 = 3|V(Q)|$ edges

Constructing the DK hierarchy

Theorem Given Q via DCEL, a DK hierarchy of

- depth $k = O(\log n)$,
- size $\sum_{i=1}^{k}(|V(Q_i)|) = O(n)$,
- and degree $\max_i \max\{\deg_{G(Q_i)}(v) \mid v \in V(Q_i) \setminus V(Q_{i+1})\} \leq 11$

can be computed in $O(n)$ time.

Proof. Iteratively remove set S, a greedy maximal independent set among vertices of degree ≤ 11.

Claim: $|S| \geq |V(Q)|/24$.
Suppose not: $|S| < |V(Q)|/24$

$\Rightarrow \bigcup_{s \in S} N[s] < |V(Q)|/2$
$\Rightarrow G(Q)$ has $\geq |V(Q)|/2$ vertices of degree ≥ 12
$\Rightarrow G(Q)$ has $\geq (|V(Q)|/2) \cdot 12/2 = 3|V(Q)|$ edges

Euler's formula:
$E(Q) \leq 3|V(Q)| - 6$

↯ □

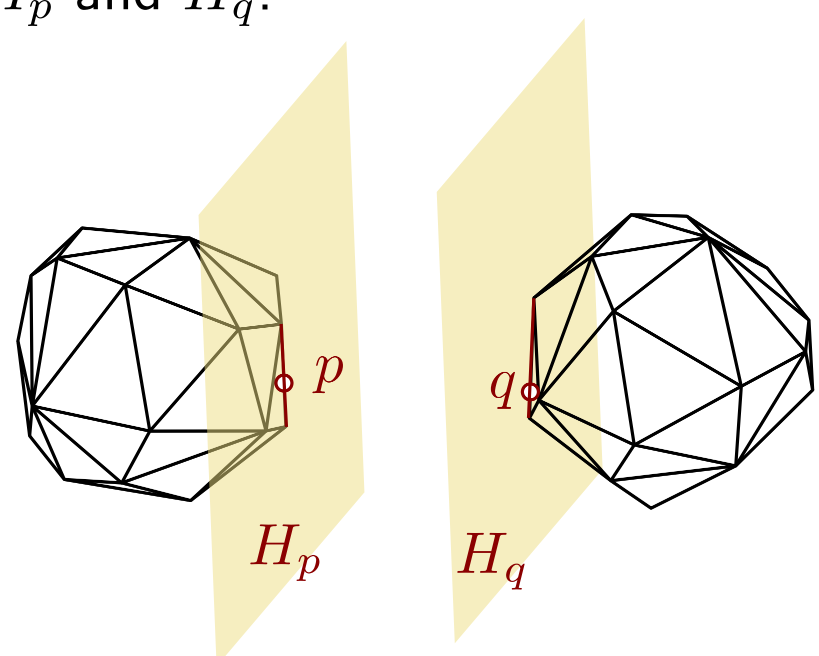
Intersection of convex polytopes in $\mathbb{R}^{\leq 3}$ via DK hierarchy

Theorem (Dobkin, Kirkpatrick '90)
Given the DK hierarchy of two convex polytopes with n and m vertices, a point in their intersection or a separating plane can be found in $O(\log n \cdot \log m)$ time.

The separating pair of P and Q is a point pair $p \in P$ and $q \in Q$ s.t.

$$\sigma(P, Q) := \min_{x \in P, y \in Q} \text{dist}(x, y) = \text{dist}(p, q)$$

p, q have parallel supporting planes H_p and H_q.



Maintaining separation via DK

Lemma
Given P with a DK-hierarchy $P_1, \ldots, P_r$ and a plane H, $\sigma(H, P)$ can be found in $O(\log n)$ time.

Sublinear intersection of convex polytopes without preprocessing

Theorem (Chazelle, Liu, Magen '06)
Given convex polyhedra P and Q by DCEL, and stored in a way that we can sample an edge from either, we can decide if P and Q intersect in $O(\sqrt{n})$ time.

Finding p_1

Sampling lemma

Ground set S, (sample) set $R \subset S$ of size r.
$\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \big\{ s \in S \setminus R \mid \varphi(R \cup \{s\} \neq \varphi(R)) \big\}$$

$$X(R) := \big\{ s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R) \big\}$$

Sampling lemma

Ground set S, (sample) set $R \subset S$ of size r.
$\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \big\{ s \in S \setminus R \mid \varphi(R \cup \{s\} \neq \varphi(R)) \big\}$$
$$X(R) := \big\{ s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R) \big\}$$

Set $v_r := \mathbf{E}(V(R))$ and $x_r := \mathbf{E}(X(R))$.

Sampling lemma

Ground set S, (sample) set $R \subset S$ of size r.
$\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \{ s \in S \setminus R \mid \varphi(R \cup \{s\} \neq \varphi(R)) \}$$
$$X(R) := \{ s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R) \}$$

Set $v_r := \mathbf{E}(V(R))$ and $x_r := \mathbf{E}(X(R))$.

Lemma(Gärtner, Welzl '01)
For $0 \leq r < n$, we have:

$$\frac{v_r}{n-r} = \frac{x_{r+1}}{r+1}.$$

Perturbing and tweaking the sampling distribution