Intersection test, ray shooting, and volume

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Geometric algorithms with limited resources
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max planck institut
informatik

Overview

- Finding an intersection point — revisited
- Testing if convex polytopes intersect with preprocessing
- Testing if convex polytopes intersect without preprocessing
- Ray shooting, nearest neighbor
- Volume approximation

Finding an intersection point revisited

Theorem (Chazelle, Liu, Magen '06)
Given two convex polygons with cyclic list of vertices, we can decide if they intersect in $O(\sqrt{n})$ expected time.

Finding an intersection point revisited

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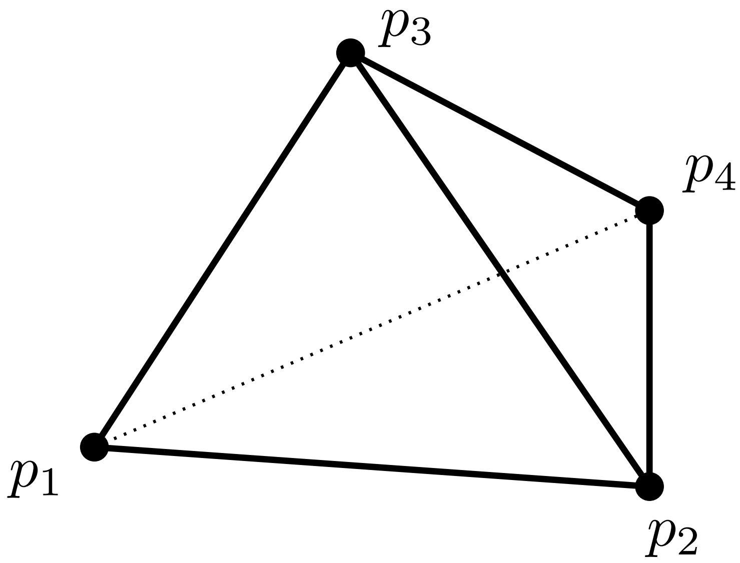
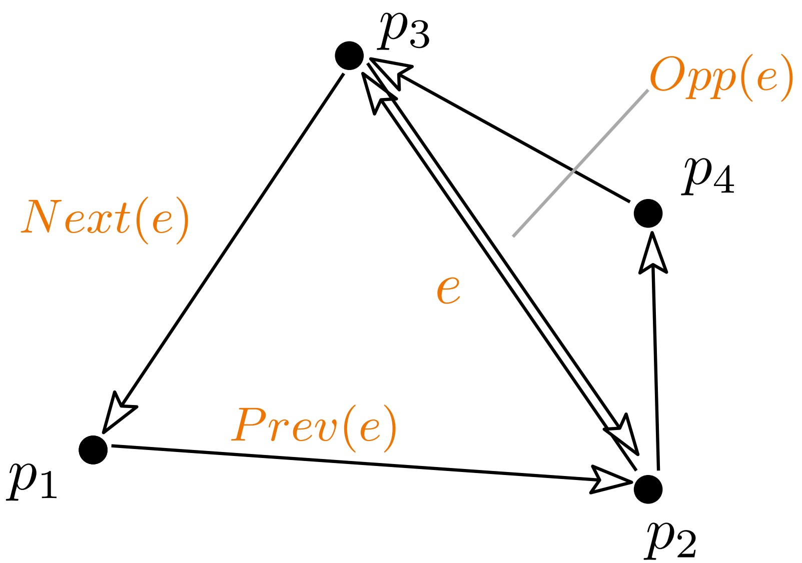
Needs: detecting separating line or intersection of sample polygons.

Convex polytope in $\mathbb{R}^3$ via DCEL





DCEL = Doubly connected edge list,
facets are ccw cycles from outside
arcs know: opposite, next, prev arc

Dobkin–Kirkpatrick hierarchy

Given convex polytope Q in $\mathbb{R}^3$, a polytope sequence $Q_1, Q_2, \ldots, Q_k$ is a DK hierarchy of Q if

1. $Q_1 = Q$ and Q_k is a tetrahedron
2. $Q_i \supset Q_{i+1}$ and $V(Q_i) \supset V(Q_{i+1})$
3. $V(Q_i) \setminus V(Q_{i+1})$ is an independent set in $G(Q_i)$.

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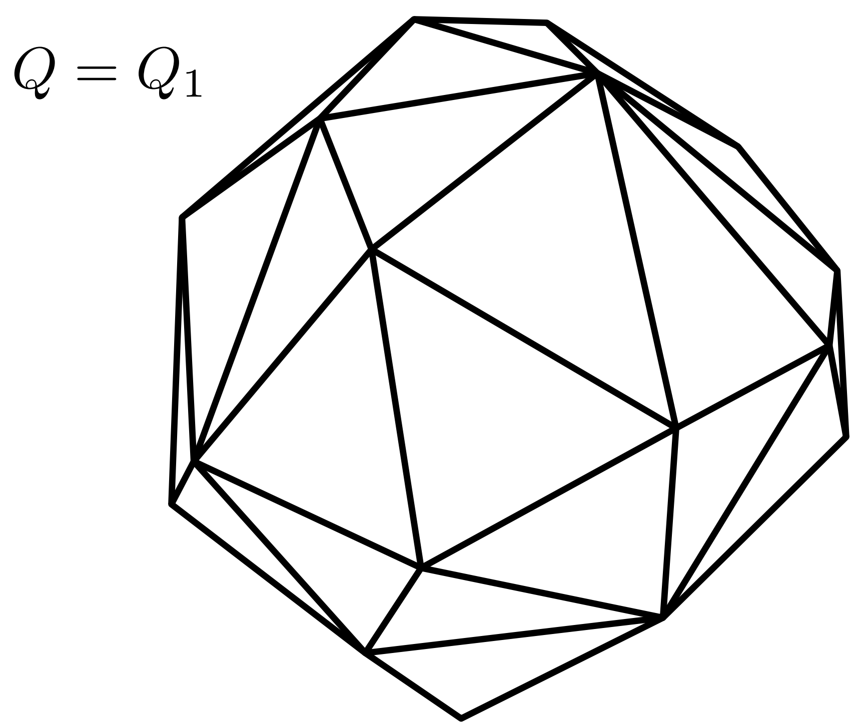

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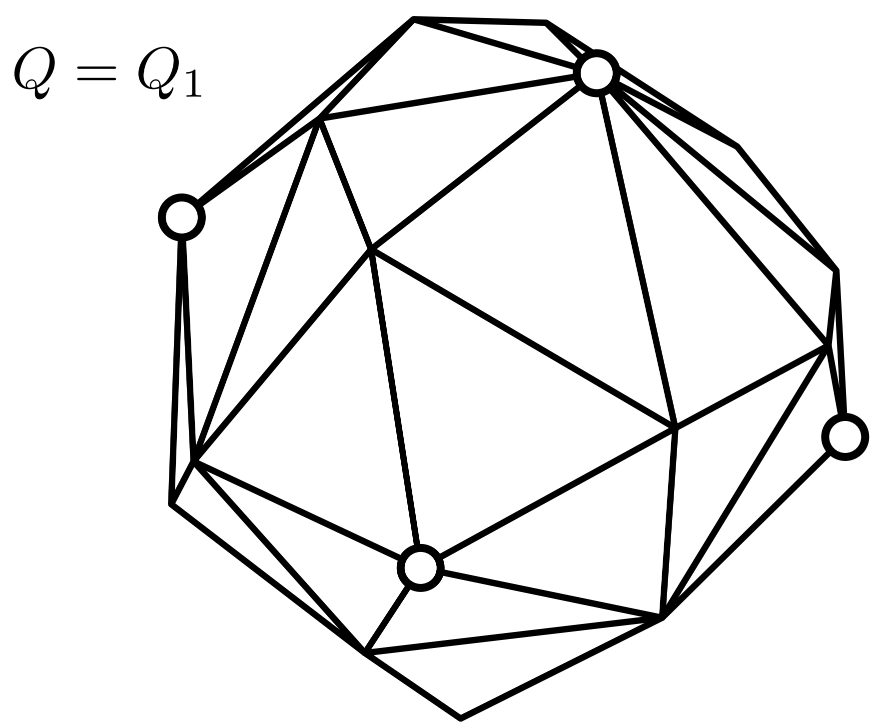

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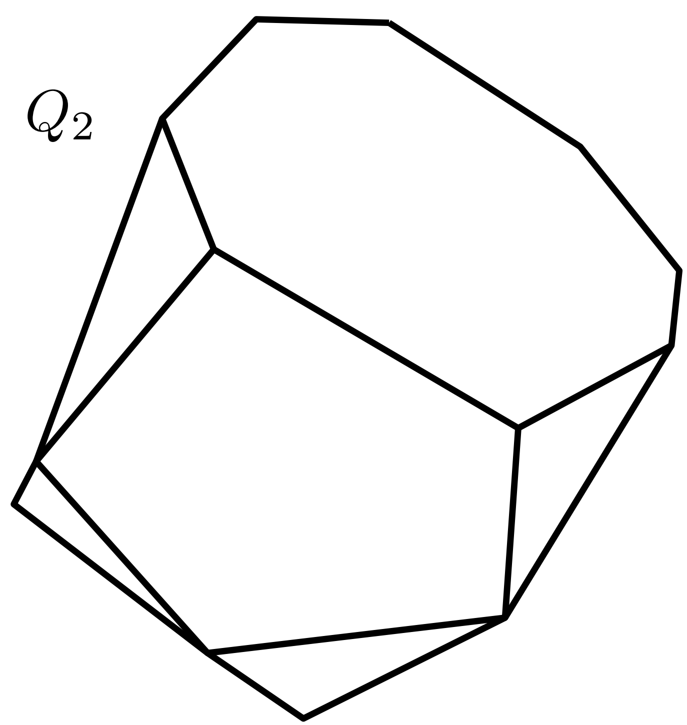

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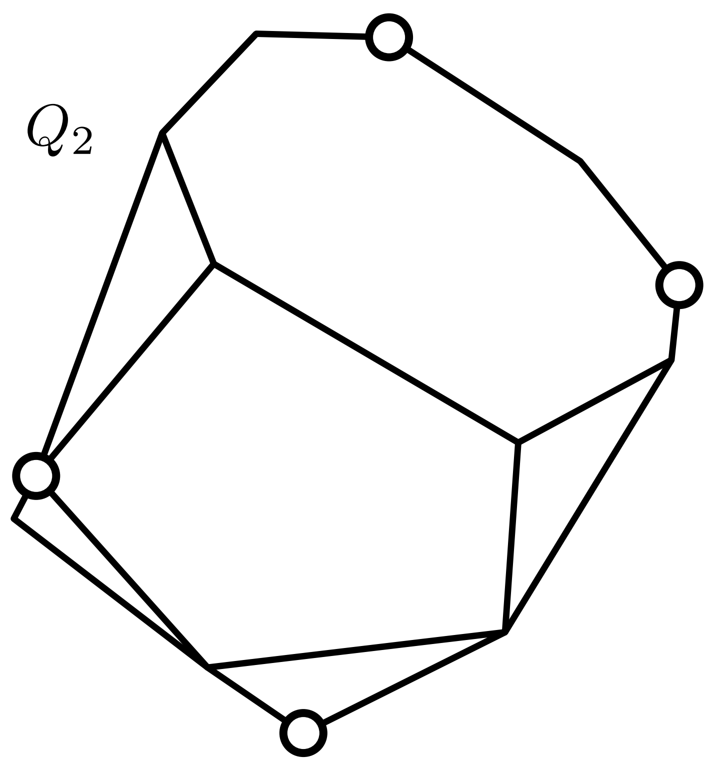

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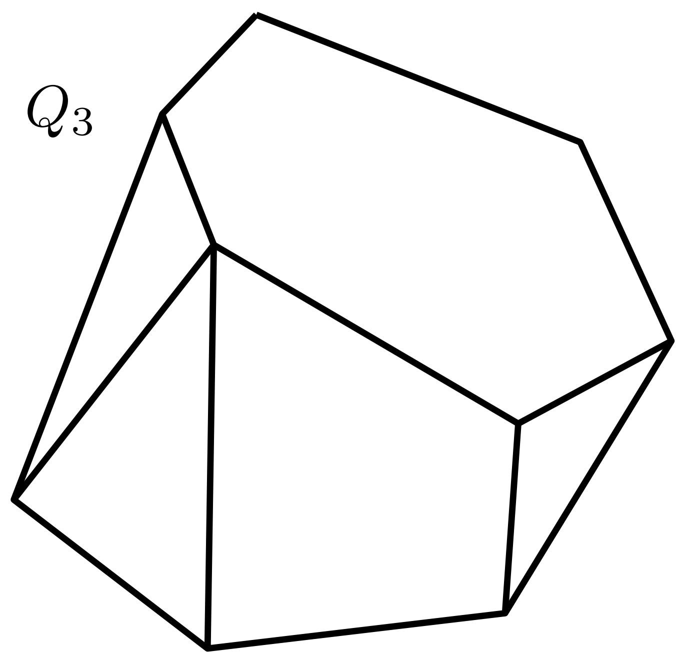

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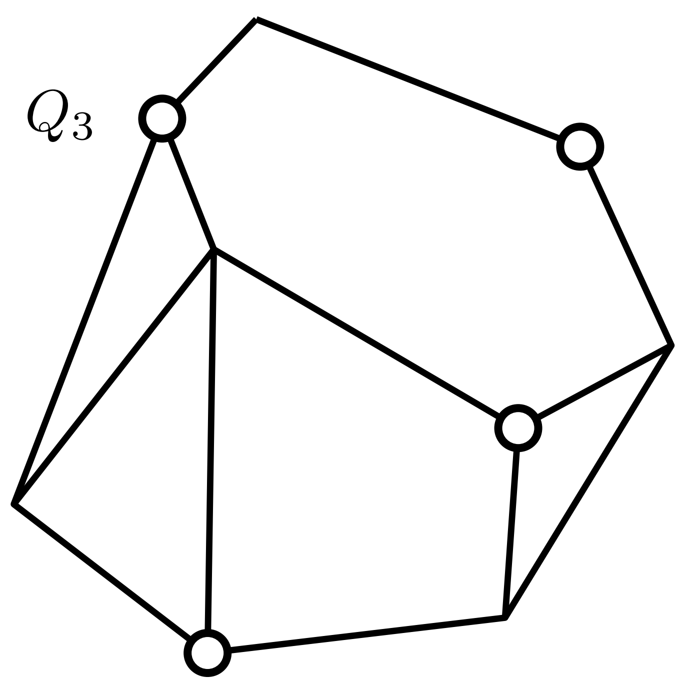

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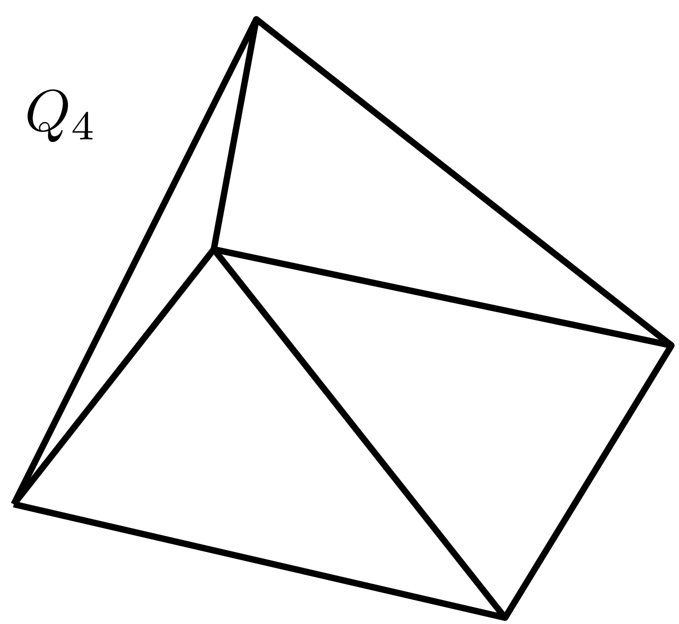

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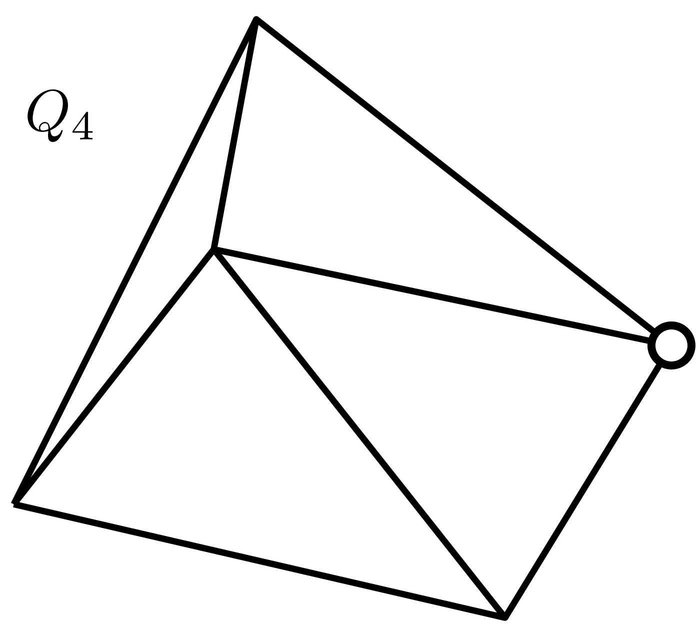

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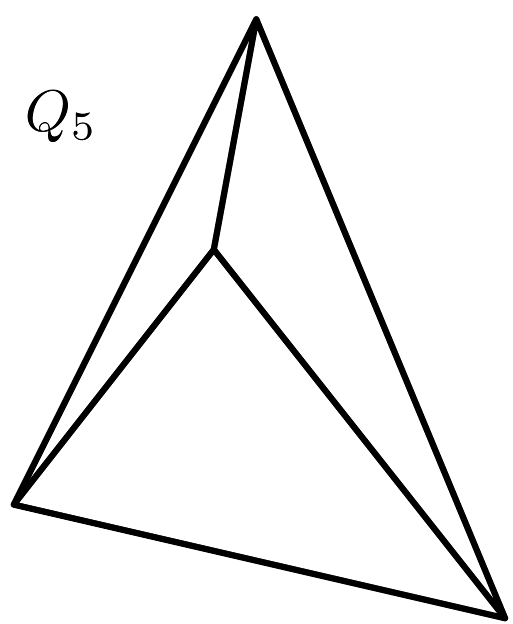

Constructing the DK hierarchy

Theorem Given Q via DCEL, a DK hierarchy of
- depth $k = O(\log n)$,
- size $\sum_{i=1}^{k}(|V(Q_i)|) = O(n)$,
- and degree $\max_i \max\{\deg_{G(Q_i)}(v) \mid v \in V(Q_i) \setminus V(Q_{i+1})\} \leq 11$

can be computed in $O(n)$ time.

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Proof. Iteratively remove set S, a greedy maximal independent set among vertices of degree ≤ 11.

Claim: $|S| \geq |V(Q)|/24$.
Suppose not: $|S| < |V(Q)|/24$
$\Rightarrow \bigcup_{s \in S} N[s] < |V(Q)|/2$
$\Rightarrow G(Q)$ has $\geq |V(Q)|/2$ vertices of degree ≥ 12
$\Rightarrow G(Q)$ has $\geq (|V(Q)|/2) \cdot 12/2 = 3|V(Q)|$ edges

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Euler's formula:
$E(Q) \leq 3|V(Q)| - 6$

↯ $\square$

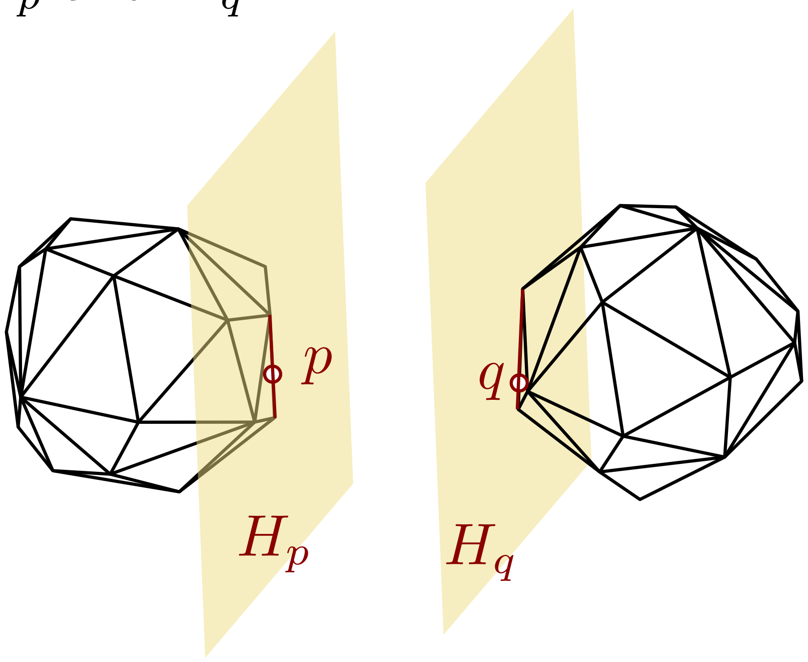
Intersection of convex polytopes in $\mathbb{R}^{\leq 3}$ via DK hierarchy

Theorem (Dobkin, Kirkpatrick '90)
Given the DK hierarchy of two convex polytopes with n and m vertices, a point in their intersection or a separating plane can be found in $O(\log n \cdot \log m)$ time.

The separating pair of P and Q is a point pair $p \in P$ and $q \in Q$ s.t.

$$\sigma(P, Q) := \min_{x \in P, y \in Q} \text{dist}(x, y) = \text{dist}(p, q)$$

p, q have parallel supporting planes H_p and H_q.



Maintaining separation via DK

Lemma

Given P with a DK-hierarchy $P_1, \ldots, P_r$ and a plane H, $\sigma(H, P)$ can be found in $O(\log n)$ time.

Sublinear intersection of convex polytopes without preprocessing

Theorem (Chazelle, Liu, Magen '06)
Given convex polyhedra P and Q by DCEL, and stored in a way that we can sample an edge from either, we can decide if P and Q intersect in $O(\sqrt{n})$ time.

Finding p_1

Sampling lemma

Ground set S, (sample) set $R \subset S$ of size r.
$\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \{ s \in S \setminus R \mid \varphi(R \cup \{s\} \neq \varphi(R)) \}$$
$$X(R) := \{ s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R) \}$$

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Set $v_r := \mathbf{E}(V(R))$ and $x_r := \mathbf{E}(X(R))$.

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Lemma(Gärtner, Welzl '01)
For $0 \leq r < n$, we have:

$$\frac{v_r}{n-r} = \frac{x_{r+1}}{r+1}.$$

Perturbing and tweaking the sampling distribution

M: multiset of vertices of $P \cup Q$, where p has $\deg(p)$ copies

$\mathcal{D}_2$: Choose $R_p \cup R_q$ by selecting each vertex of M indep. with prob. r/n

Ray shooting, Voronoi pt location

Theorem
Given a convex polytope (as DCEL) of n vertices and a directed line, their intersection can be computed in $O(\sqrt{n})$ time.

Theorem
Given a Delaunay triangulation or a Voronoi diagram as DCEL, we can compute point location (i.e., identify the cell a given query point falls into) in $O(\sqrt{n})$ time.

$$p = (p_x, p_y) \rightarrow \ H_p : z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Nearest point of a polytope

$n_P(q)$: nearest point of P to q
$\xi_P(\ell)$: point of largest ℓ-coordinate in P
$\xi_P(H, \ell)$: point of largest ℓ-coordinate in $P \cap H$

Theorem
Given a convex polytope P (as DCEL) of n vertices, a point q and a directed line ℓ, we can compute $n_P(q), \xi_P(\ell), \xi_P(H, \ell)$ in $O(\sqrt{n})$ time.

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(n/\varepsilon)$ time.

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Stage 1. Reshaping into ball-like polytope

Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope Q s.t. $P \subset Q \subset P_\varepsilon$ by projecting $(1/\sqrt{\varepsilon})$-net of sphere