Ray shooting and volume approximation

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Overview

- Testing if convex polytopes intersect without preprocessing – wrap-up
- Ray shooting, nearest neighbor
- Volume approximation

Sublinear intersection of convex polytopes without preprocessing

Theorem (Chazelle, Liu, Magen '06)
Given convex polyhedra P and Q by DCEL, and stored in a way that we can sample an edge from either, we can decide if P and Q intersect in $O(\sqrt{n})$ time.

Recall:

- sample from both of size $r = \sqrt{n}$
- find separating plane H (if not, return intersection)
- $p \in H \cap V(P)$ has neighbor p_1 on other side, find it by resampling

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Last time:
Ground set S, (sample) set $R \subset S$ of size r. $\varphi : 2^S \to \mathbb{R}$ Let

$$V(R) := \{s \in S \setminus R \mid \varphi(R \cup \{s\} \neq \varphi(R))\}$$

$$X(R) := \{s \in R \mid \varphi(R \setminus \{s\}) \neq \varphi(R)\}$$

Set $v_r := \mathbf{E}(V(R))$ and $x_r := \mathbf{E}(X(R))$.

Sampling Lemma(Gärtner, Welzl '01)
For $0 \leq r < n$, we have:

$$\frac{v_r}{n-r} = \frac{x_{r+1}}{r+1}.$$

Perturbing and tweaking the sampling distribution

M: multiset of vertices of $P \cup Q$, where p has $\deg(p)$ copies

M': perturb M by moving infinitesimally randomly towards edge midpoints

$\mathcal{D}_3$: Choose $R_p \cup R_q$ by selecting each vertex of M' indep. with prob. r/n

$\mathcal{D}_2$: Choose $R_p \cup R_q$ by selecting each vertex of M indep. with prob. r/n

Ray shooting, Voronoi pt location

Theorem
Given a convex polytope (as DCEL) of n vertices and a directed line, their intersection can be computed in $O(\sqrt{n})$ time.

Theorem
Given a Delaunay triangulation or a Voronoi diagram as DCEL, we can compute point location (i.e., identify the cell a given query point falls into) in $O(\sqrt{n})$ time.

$$p = (p_x, p_y) \to \ H_p : z = 2p_x x + 2p_y y - (p_x^2 + p_y^2)$$

Nearest point of a polytope

$n_P(q)$: nearest point of P to q
$\xi_P(\ell)$: point of largest ℓ-coordinate in P
$\xi_P(H, \ell)$: point of largest ℓ-coordinate in $P \cap H$

Theorem
Given a convex polytope P (as DCEL) of n vertices, a point q and a directed line ℓ, we can compute $n_P(q), \xi_P(\ell), \xi_P(H, \ell)$ in $O(\sqrt{n})$ time.

Volume approximation

Theorem
Given $\varepsilon > 0$ and a convex polyope P on n vertices, we can compute a $(1+\varepsilon)$-approximation of its volume in $O(\sqrt{n}/\varepsilon)$ time.

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Stage 2. Coreset-like approximation with $O(1/\varepsilon)$ size polytope Q s.t. $P \subset Q \subset P_\varepsilon$ by projecting $(1/\sqrt{\varepsilon})$-net of sphere

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Stage 1 will use:

Theorem. Any compact convex object $K \subset \mathbb{R}^d$ has a unique maximum volume ellipsoid $\mathcal{E} \subseteq K$.

Theorem (John 1948). For any compact convex $K \subset \mathbb{R}^d$ with $\mathcal{E}$ centered at the origin, $\mathcal{E} \subseteq K \subseteq d\mathcal{E}$.