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Techniques for Counting Problems, Exercise Sheet 1

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer23/counting

Total Points: **100** Due: Thursday, **April 27**, 2023

You are allowed to collaborate on the exercise sheets. Justify your answers. Cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Please hand in your solutions before the lecture on the day of the deadline.

Exercise 1 — (5 + 5 + 5) + 10 points

We say that a Boolean formula is in k conjunctive normal form (k-CNF) if it is the conjunction of clauses where each clause is a disjunction of *at most* k literals (that is, possibly negated variables).

a Express each of the following Boolean formulas as a 3-CNF formula (possibly using new variables):

$$\text{(i) } c = a \wedge b \qquad \text{(ii) } c = a \to b \qquad \text{(iii) } d = (a \vee b) \wedge \overline{c}$$

b Express $x_1 \to \big((x_2 \wedge x_3) \vee (\overline{x_2} \wedge x_4)\big)$ as a 3-CNF formula with the same number of satisfying assignments

Exercise 2 — 15 + 10 points

In the DNF-SAT problem, we are given a Boolean formula in disjunctive normal form (DNF) and the task is to decide whether the formula is satisfiable. We write #DNF-SAT for the corresponding counting problem.

a Prove that DNF-SAT is in P.

b Prove that #DNF-SAT is #P-complete.

Exercise 3 — 10 + 5 + 15 points

In the Max-2-SAT problem, we are given a Boolean formula in 2-CNF along with an integer k and the task is to check whether there is an assignment that satisfies at least k clauses of the given formula.

a Let ϕ denote a 3-CNF formula with m clauses. Suppose that we replace every clause of the form $(a \vee b \vee c)$ by the following clauses

$$(a) \wedge (b) \wedge (c) \wedge (d) \wedge (\overline{a} \vee \overline{b}) \wedge (\overline{a} \vee \overline{c}) \wedge (\overline{b} \vee \overline{c}) \wedge (a \vee \overline{d}) \wedge (b \vee \overline{d}) \wedge (c \vee \overline{d}).$$

Explain how (and why) this construction can be used to obtain a reduction from 3-SAT to Max-2-SAT.

b Prove that the reduction from a) is not parsimonious.

c Show a parsimonious reduction from #3-SAT to #Max-2-SAT.

Hint: Introduce two new variables for each clause of the original formula. There is a solution that replaces original clauses with at most 30 clauses each.

Exercise 4 — 10 + 10 points

For Max-2-SAT one can duplicate clauses to "simulate" a weight, that is, it is more important to satisfy the repeated clauses than others. However, for certain reductions it might be necessary to avoid duplicate clauses.

a Describe an efficient reduction to replace duplicate clauses in a way that is *not* parsimonious.

b Describe an efficient reduction to replace duplicate clauses in a way that *is* parsimonious.

Hint: Introduce two new variables per duplicate clause.