Jacob Focke and Philip Wellnitz Summer 2023

Techniques for Counting Problems, Exercise Sheet 3

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/summer23/counting

Total Points: **100** Due: Thursday, **May 25**, 2023

You are allowed to collaborate on the exercise sheets. Justify your answers. Cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Please hand in your solutions before the lecture on the day of the deadline.

Exercise 1 — **20** points

For any two graphs A, B (possibly with loops but without multiedges) and a simple graph G, show that

$$\mathsf{hom}(G, A \otimes B) = \mathsf{hom}(G, A) \cdot \mathsf{hom}(G, B).$$

Exercise 2 — **30** points

Recall that #VC is the problem of counting all vertex covers of *any* size in a simple graph.

Let #CardinalityVC be the problem that takes as input a simple graph G together with a non-negative integer k and asks to output the number of vertex covers of G of size k.

Give a polynomial-time Turing-reduction from #CardinalityVC to #VC.

Hint: Use a graph $G(\ell)$, which is obtained from G by introducing ℓ copies of each vertex and introducing an edge between copies whenever the original vertices shared an edge. For each cover of G define a sensible class of corresponding covers in $G(\ell)$. Then express the number of covers of $G(\ell)$ using a partitioning that depends on the size of a corresponding cover of the original graph G.

Exercise 3 — **20** points

Let B be a bipartite graph. For each vertex v of B, let B_v be the subgraph of B induced by all vertices of B that have distance at most 2 from v (including v).

Moreover, let G be a connected bipartite graph with a bipartition (L, R). Then let G_L^a be the graph obtained by adding a vertex a to G that is completely connected to R (so one can think of a being in L). Similarly, let G_R^a be the graph obtained by adding a vertex a to G that is completely connected to L. Assume without a proof that the following statement holds:

$$\mathsf{hom}(G_L^a, B) + \mathsf{hom}(G_R^a, B) = \sum_{v \in V(B)} \mathsf{hom}(G, B_v).$$

Show that, for every vertex v of B, there is a polynomial-time Turing reduction from #Hom(B_v) to #Hom(B).

Hint: Use complexity monotonicity.

Exercise 4 — **25 + 5** points

Recall that a connected graph is hom-easy if it is a looped complete graph or a complete bipartite graph. In general, a graph is hom-easy if each connected component is hom-easy.

You can assume the following two results without a proof.

- Given positive integers a_1, a_2, a_3, a_4, let $P_4(a_1, a_2, a_3, a_4)$ be the graph that consists of four independent sets of vertices $A_1, \dots, A_4$ of size $a_1, \dots, a_4$, respectively, such that for $i = 1, 2, 3$ the vertices in A_i are completely connected to A_{i+1}. (Basically, this is a four-vertex path with a_1 copies of the first vertex, a_2 copies of the second vertex, and so on.)

For all fixed $a_1 \ldots, a_4$, the problem $\#\textsc{Hom}(P_4(a_1, a_2, a_3, a_4))$ is $\#\mathsf{P}$-complete.

- Let B be a connected bipartite graph such that, for each vertex v of B, the graph B_v (as defined in the previous exercise) is either a complete bipartite graph or otherwise is B itself.

  If B is not a complete bipartite graph, then B is isomorphic to $P_4(a_1, \ldots, a_4)$ for some positive integers a_1, a_2, a_3, a_4.

- If H has a connected component that is not hom-easy then $H \otimes K_2$ has a connected component that is not hom-easy.

Use the results from the lecture and the ones above to show the following, that is, the main hardness result for counting homomorphisms with fixed right-hand side.

a For any bipartite graph B that is not hom-easy, $\#\textsc{Hom}(B)$ is $\#\mathsf{P}$-complete.

b For any graph H that is not hom-easy, $\#\textsc{Hom}(H)$ is $\#\mathsf{P}$-complete.