An Improved Distributed Algorithm for Maximal Independent Set (MIS)

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Model: distributed LOCAL model -

all nodes learn in k-rounds the subgraph induced by their k-hop neighborhood (+ random bits)

Maximal Independent Set

For a graph $G=(V,E)$ an independent set S (the subgraph induced on the set is empty) is a maximal independent set if $\forall v \in V$: either $v \in S$ or $N(v) \cap S \neq \emptyset$.

Result (main): MIS in $O(\log \Delta) + 2^{O(\sqrt{\log \log n})}$ (randomized)

improving Barenboim et al. $O(\log^2 \Delta) + 2^{O(\sqrt{\log \log n})}$

best lower bound: $\Omega\left(\frac{\log \Delta}{\log \log \Delta}\right)$ and $\Omega\left(\sqrt{\frac{\log n}{\log \log n}}\right)$ Kuhn et al.

high level overview of the Alg.

Step 1: run $O(\log \Delta + \log 1/\varepsilon)$ rounds of "local algorithm".
s.t. for each node v, w.p $\geq 1-\varepsilon$ v is in the MIS or one of its neighbors is.

Step 2: w.h.p the remaining graph is shattered into small components, for which we can solve MIS deterministically. $\rightarrow$ the $2^{O(\sqrt{\log \log n})}$ term.

The "Local" algorithm.

In each round t:

$\forall v \in V$ $\quad P_t(v)$ - the desired level of v

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v gets marked w.p. $P_t(v)$,

if none of v's neighbors is marked

then v joins the MIS and we remove $N^+(v)$ from G.

$$N^+(v) \equiv \{v\} \cup N(v)$$

$$\boxed{\begin{aligned} & P_0(v) = \tfrac{1}{2} \\ & d_t(v) = \sum_{u \in N(v)} P_t(u) \quad \text{- effective degree} \\ & P_{t+1}(v) = \begin{cases} P_t(v)/2 & \text{if } d_t(v) \geq 2 \\ \min\{2P_t(v), 1/2\} & \text{if } d_t(v) < 2 \end{cases} \end{aligned}}$$

Thm 1: Fix $v \in V$ by round $X = \beta(\log \Delta + \log 1/\varepsilon)$

v $\underbrace{\text{made its decision}}_{v \in MIS \text{ or } N(v) \cap MIS \neq \emptyset}$ w.p. $\geq 1-\varepsilon$.

golden rounds of v:

[v has a good chance to join the MIS]

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type 1: $d_t(v) < 2$ and $P_t(v) = \frac{1}{2}$

type 2: $d_t(v) \geq 1$ and at least $\frac{d_t(v)}{10}$ of it from low-degree neighbors $(d_t < 2)$

[v has a good chance that one of its neighbors joins the MIS]

Lemma (easy proof): in golden rounds v has constant pr. of being removed from G.

$\Rightarrow$ to prove thm 1, suffices to show that a constant fractions of the rounds are golden

proof: g_1 - # golden rounds of type 1.

g_2 - # golden rounds of type 2.

ask the class

Assume $g_1 \leq c \cdot X$ (otherwise we are done).

define h = #[rounds $d_t(v) \geq 2$] = #[rounds $P_t(v)$ dec.] $\geq$ #[rounds $P_t(v)$ inc.]

$\Rightarrow$ $P_t(v)$ inc or dec. $\leq 2h$ rounds

$\Rightarrow$ $\geq X-2h$ rounds $P_t(v) = \frac{1}{2}$

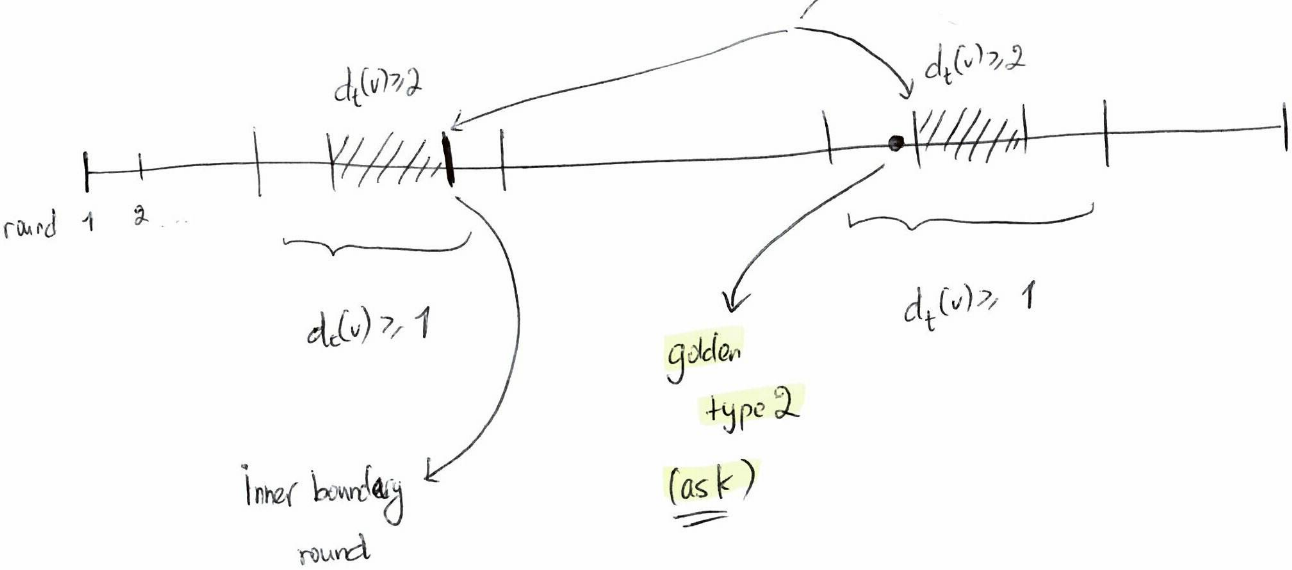
$\Rightarrow$ $\geq X-3h$ rounds $P_t(v) = \frac{1}{2}$ and $d_t(v) < 2$ (golden type 1)

$g_1 \geq X-3h$ and so $h \geq \frac{(1-c)}{3} X$ (const. fraction of X).

Lemma: If $d_t(v) \geq 1$ and t not golden type 2; then $d_{t+1}(v) < \frac{2}{3} d_t(v)$

proof: $d_{t+1}(v) \leq 2 \cdot \frac{1}{10} d_t(v) + \frac{1}{2} \frac{9}{10} d_t(v) < \frac{2}{3} d_t(v)$

focus on rounds for which $d_t(v) \geq 2$:



A - rounds ~~from~~ for which $d_t(v) \geq 2$

I - rounds in A which are inner-boundary. ($d_t(v)$ incr. at most factor 2)

observe $|I| \leq g_2$

the rounds in $A \setminus I$ can be golden or non-golden.

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a = # golden rounds in $A \setminus I$ ($d_t(v)$ incr. by at least factor 2)

b = # non-golden rounds in $A \setminus I$. ($d_t(v)$ dec. by at least factor $2/3$)

$\cancel{|A| = |I| + a + b} \;\; \cancel{g_2 + a + b}$

Lemma*: ~~a th~~ $b \leq 2|I| + 2a + \log_{3/2} \Delta$

proof: assume otherwise and obtain that $d_t(v) < 2$.

use: 1) $(2/3)^2 \cdot 2 < 1$

2). $d_t(v) \leq \frac{\Delta}{2} \quad \forall t$.

3) $\frac{\Delta}{2} \cdot \left(\frac{2}{3}\right)^{\log_{3/2} \Delta} < 2$

Since $|A| = h = |I| + a + b \le g_2 + a + b$,

we obtain from Lemma * that:

$$a \ge h - g_2 - b \ge h - g_2 - \left(2|I| + 2a + \log_{3/2} \Delta\right)$$

$$\ge h - 3g_2 - 2a - \log_{3/2} \Delta$$

$$(6 g_2 / \ge) \quad 3a + 3g_2 \ge h - \log_{3/2} \Delta$$

and so $\quad 6 g_2 \ge h - \log_{3/2} \Delta$

$$g_2 \ge \frac{h - \log_{3/2} \Delta}{6}, \quad g_2 \text{ constant fraction of } X.$$

$\square$

Shattering facts:

1) in each round, v decides wether to join the MIS based only on random coins of $N_v^1(v)$ $\Rightarrow$ there is a lot of independence

2) By Pancorosi and Srinivasan, deterministic MIS on m nodes $2^{O(\sqrt{\log m})}$.