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Exercises for Fine-Grained Complexity Theory

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter17/fine-complexity/

Exercise sheet 2 Due: **Tuesday, November 21, 2017**

Total points : 40 + 10 bonus points

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words.*** *Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets.

Exercise 1 (*12 points*) Recall that in the lecture, we generalized **OV** to the following problem:

k–**OV**: Let k sets $A_1, A_2, \ldots, A_k \subseteq \{0,1\}^d$ with $|A_1| = |A_2| = \ldots = |A_k| = n$ be given. Decide whether there exist $a^{(1)} \in A_1, a^{(2)} \in A_2, \ldots, a^{(k)} \in A_k$ such that in every dimension the corresponding component of at least one of the vectors $a^{(1)}, a^{(2)}, \ldots, a^{(k)}$ is 0.

Consider the following hypothesis about this family of problems:

kOVH: For no $k \geq 2$ and $\varepsilon > 0$, there is an algorithm for k–**OV** running in time $O(n^{k-\varepsilon} \cdot \text{poly}(d))$.

a) *(6 points)* In the lecture, we introduced the *q–Dominating Set* problem:

   q–**DomSet**: Given a graph $G = (V, E)$, decide whether there is a subset of the vertices $S \subseteq V$ of size q, such that for any vertex $v \in V$, either $v \in S$ or $\{u, v\} \in E$ for some $u \in S$.

   Prove that q–**DomSet** cannot be solved in time $O(n^{q-\varepsilon})$ for all $\varepsilon > 0$ and integers $q \geq 3$, unless **kOVH** fails.

b) *(6 points)* Consider the following variant of **kOVH**:

   kOVH′: For no $k \geq 100$ and $\varepsilon > 0$, there is an algorithm for k–**OV** running in time $O(n^{k-\varepsilon} \cdot \text{poly}(d))$.

   Show that **kOVH** and **kOVH′** are equivalent.

Exercise 2 (*15 points*) In this exercise, we will prove further results about **OV**.

a) *(5 points)* Give an algorithm for **OV** running in time $\tilde{O}(n^2) = O(n^2 \cdot \text{poly}\log(n))$ for vectors of dimension $d = n^{0.1}$.

b) *(5 points)* Show that if **OV** can be solved in time $T(n, d)$, then given any **OV** instance we can also *find* an orthogonal pair, if it exists, in time $O(T(n, d))$.

c) *(5 points)* Adapt the **OV** algorithm from the lecture to also *find* an orthogonal pair, if it exists, in the same asymptotic running time of $n^{2-1/O(\log c)}$. *(Recall that for the algorithm from the lecture, the vectors have a dimension of $d = c \cdot \log(n)$ with $c = n^{o(1)}$.)*
Your solution to this exercise must be different from your solution of part b) above.

Exercise 3 (*13 points + 10 bonus points*) Recall the *Longest Common Substring With Don't Cares* problem from the previous exercise sheet:

Longest Common Substring With Don't Cares: Given a string A of length n over some alphabet Σ and string B of length n over the alphabet $\Sigma \cup \{\star\}$, find the length $L(A, B)$ of the longest string that is a substring of both A and B, where a "$\star$" in B can be treated as any character from the alphabet Σ.

In this exercise, we consider only the binary alphabet $\Sigma = \{0, 1\}$.

a) *(8 points)* Let strings $A \in \{0, 1\}^n, B \in \{0, 1, \star\}^n$ be such that their longest common substring has length $L(A, B) \leq c \cdot \log(n)$ with $c = n^{o(1)}$.
Show how to compute the length $L(A, B)$ of the longest common substring of A and B in time $n^{2-1/O(\log c)}$.

b) *(*10 **bonus** points*)* Given strings $A \in \{0, 1\}^n, B \in \{0, 1, \star\}^n$ and $\Delta \in \mathbb{N}$. Show how to determine whether the longest common substring of A and B has a length of at least Δ, i.e., whether $L(A, B) \geq \Delta$, and if so, how to compute $L(A, B)$; both in time $O(n^2/\sqrt{\Delta})$.

*Hint 1: You may assume you can solve the following problem in time $O(n \log m)$: Given a text $T \in \{0, 1\}^n$ and a pattern $P \in \{0, 1, \star\}^m$ (with wildcards), determine **all** occurrences of P in T, i.e., all indices $1 \leq i \leq n - m + 1$ such that $T[i..i + m - 1]$ and P match.*

Hint 2: Use the following approach: Divide T, P into blocks and try to find completely matching block pairs to establish a good lower bound on L. Once you were successful, try to extend matching substrings as much as possible.

c) *(5 points)* Show that **Longest Common Substring With Don't Cares** can be solved in time $n^2/2^{\Omega(\sqrt{\log n})}$.

(Note: You can use a) and b) even if you didn't solve them.)