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Exercises for Fine-Grained Complexity Theory

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter17/fine-complexity/

Exercise Sheet 4 Due: **Tuesday, December 19, 2017**

Total points : 40

*You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, **using your own words.** Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets.

Exercise 1 (*10 points*) Recall the formal definition of fine-grained reductions from the lecture:

For problems P, Q and (conjectured) time bounds T_P, T_Q for these problems, the pair (P, T_P) has a *fine-grained reduction* to the pair (Q, T_Q), denoted by $(P, T_P) \leq_{fgr} (Q, T_Q)$, if and only if for any $\varepsilon > 0$ there are a $\delta > 0$ and a word RAM machine M^Q (with oracle access to Q) such that, given an input of size n,

- the machine M^Q solves the problem P in time $O(T_P(n)^{1-\delta})$, and
- the calls to the oracle on the corresponding oracle inputs $I_1, I_2, \ldots, I_k$ of sizes $n_1 = |I_1|, n_2 = |I_2|, \ldots, n_k = |I_k|$ satisfy $\sum_{i\in[k]} T_Q(n_i)^{1-\varepsilon} \leq O(T_P(n)^{1-\delta})$.

Prove that fine-grained reductions are transitive, i.e., prove that for any problems P, Q, and R and corresponding time bounds T_P, T_Q, and T_R, if there are fine-grained reductions $(P, T_P) \leq_{fgr} (Q, T_Q)$ and $(Q, T_Q) \leq_{fgr} (R, T_R)$, then there is also a fine-grained reduction $(P, T_P) \leq_{fgr} (R, T_R)$.

Exercise 2 (*10 points*) Consider the following problem and the corresponding conjecture:

Hitting Set Problem: Given two lists of n subsets over a universe U of size d, determine if there is a set in the first list that intersects every set in the second list, i.e. a "hitting set".
Hitting Set Hypothesis (HSH): The **Hitting Set Problem** cannot be solved in time $O(n^{2-\epsilon} \cdot \text{poly}(d))$.

Prove that **HSH** implies **OVH**.

(Hint: In the lecture, we showed a reduction from All-Pairs-Negative-Triangle to Negative-Triangle. The same kind of reduction can work here.)

Exercise 3 (*4 points*) In the lecture we defined the **Negative Triangle Problem** on general graphs as follows:

Negative Triangle: Given a weighted directed graph $G = (V, E, w)$, $|V| = n$ with edge weights $w : E \to \{-n^c, \dots, n^c\}$ (for some $c > 0$), determine if there are three vertices i, j, k such that $w(i, j) + w(j, k) + w(k, i) < 0$ holds.

Consider the following variant of that problem, where G is required to be triparte:

Negative Triangle′: Given a weighted, directed, and tripartite graph $G = (A \cup B \cup C, E, w)$, $|A| = |B| = |C| = n$, with edge weights $w : E \to \{-n^c, \dots, n^c\}$ (for some $c > 0$), determine if there are three vertices $i \in A, j \in B, k \in C$ such that $w(i, j) + w(j, k) + w(k, i) < 0$ holds.

Prove that these variants are equivalent under (subcubic) fine-grained reductions, i.e., prove

$$(\textbf{NegativeTriangle}', n^3) \leq_{fgr} (\textbf{NegativeTriangle}, n^3) \leq_{fgr} (\textbf{NegativeTriangle}', n^3).$$

Exercise 4 (*6 points*) The **Metricity Problem** is defined as follows: Given an $n \times n$ matrix A with entries in $\{0, \dots, n^c\}$ for some constant $c > 0$, decide whether for every $1 \leq i, j, k \leq n$ $A_{ij} \leq A_{ik} + A_{kj}$ holds.

Prove that the **Metricity Problem** is equivalent to **APSP** under (subcubic) fine-grained reductions, i.e., prove

$$(\textbf{APSP}, n^3) \leq_{fgr} (\textbf{Metricity}, n^3) \leq_{fgr} (\textbf{APSP}, n^3).$$

(Hint: Solve Metricity using Min-Plus Product and reduce Negative Triangle(′) to Metricity.)

Exercise 5 (*10 points*) Consider the following graph problem of finding a triangle of *zero weight*:

ZeroTriangle: Given a weighted directed graph $G = (V, E, w)$ with edge weights $w : E \to \{-n^c, \dots, n^c\}$ (for some $c > 0$), determine if there are three vertices i, j, k such that $w(i, j) + w(j, k) + w(k, i) = 0$ holds.

a) *(5 points)* Given a B-bit integer x, define $pre_\ell(x)$ as the integer obtained from x by removing the last $B - \ell$ bits of x. (If you are familiar with the "right-shift operator" $\gg$ of common programming languages, then $pre_\ell(x)$ can be defined as $pre_\ell(x) := x \gg (B - \ell)$.)

   Prove that for any non-negative integers x, y, z, we have the following equivalence:

   $$x + y > z \iff \text{There are } 1 \leq \ell \leq B, b \in \{1, 2, 3\} \text{ with } pre_\ell(x) + pre_\ell(y) = pre_\ell(z) + b.$$

b) *(5 points)* Prove that there is a (subcubic) fine-grained reduction from **APSP** to **ZeroTriangle**, i.e., prove

   $$(\textbf{APSP}, n^3) \leq_{fgr} (\textbf{ZeroTriangle}, n^3).$$

   (Hint: You may reduce from Negative Triangle′ defined in Exercise 3.)