max planck institut informatik

SAARLAND UNIVERSITY
COMPUTER SCIENCE

Karl Bringmann and Marvin Künnemann **Winter 2017/18**

Exercises for Fine-Grained Complexity Theory

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter17/fine-complexity/

Exercise Sheet 6 (Final) Due: **Tuesday, January 23, 2018**

Total points : 40 + 5 bonus points

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, ***using your own words.*** *Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.).*

You need to collect at least 50% of all points on exercise sheets.

Recall the definition of co-nondeterministic algorithms from the lecture:

A decision problem P is in co-nondeterministic time $t(n)$ if there is a deterministic algorithm V_{no}, that runs on an input of size n in time $t(n)$ such that

- for any "YES"-instance I of P and for every string π, $V_{no}(I, \pi)$ rejects and
- for any "NO"-instance I of P there is a string π for which $V_{no}(I, \pi)$ accepts.

Exercise 1 (*9 points*) Recall the k–**SUM** problem from before:

k–**SUM:** Given k sets $A_1, A_2, \ldots, A_k$ of n non-negative integers and a target t, determine whether there are $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_k \in A_k$ such that $a_1 + a_2 + \ldots + a_k = t$.

Prove that for any constant $\varepsilon > 0$, there is a constant $\delta > 0$, such that for any constant $k > 3$, k–**SUM** has no $O(t^{1-\varepsilon} \cdot n^{\delta k})$ algorithm, unless **SETH** fails.

Hint: You may use that for any constant $\varepsilon > 0$, there is a constant $\delta > 0$, such that ***Subset Sum*** *has no $O(t^{1-\varepsilon} \cdot 2^{\delta n})$ algorithm, unless* ***SETH*** *fails. Design a reduction from* ***Subset Sum*** *to k–**SUM**.*

Exercise 2 (*5 points*) Recall from the lecture that a set S of integers is k-sum-free, if any k numbers $x_1, \ldots, x_k$ from S that sum up to $k \cdot \bar{x}$, for $\bar{x} \in S$, must be equal, i.e.,

$$x_1 + x_2 + \ldots + x_k = k \cdot \bar{x} \implies x_1 = \ldots = x_k = \bar{x}.$$

We proved that for $U = n^{1+\varepsilon} \cdot k^{24/\varepsilon}$, there is a k-sum-free set $S \subseteq \{1, \ldots, U\}$ of size n.

What is the smallest universe size that you can obtain if k is equal to three? In other words, find the smallest value of $n^{1+\varepsilon} \cdot k^{24/\varepsilon}$ for $k = 3$ by determining the best choice for ε.

Exercise 3 (*15 points*)

a) *(6 points)* Given a weighted graph $G = (V, E, w)$ and a prime p, demonstrate an algorithm running in time $O(p^2 \cdot n^\omega)$ that computes the number of triples $(i, j, k) \in V^3$ such that

$$w(i, j) + w(j, k) + w(k, i) = 0 \pmod p.$$

b) *(5 points)* Demonstrate nondeterministic and co-nondeterministic algorithms running in time $O(n^{3-\varepsilon})$ for **ZeroTriangle** (for some $\varepsilon > 0$).

Hint: Use part a); the (co-nondeterministic) ***3SUM*** *algorithm from the lecture is similar.*

c) *(4 points)* Consider the following decision problem:

$(\min, +)$–**Verification**: Given $n \times n$ matrices A, B, C, determine whether $A \odot B = C$, where $\odot$ denotes the $(\min, +)$–product of two matrices.

Demonstrate nondeterministic and co-nondeterministic algorithms running in time $O(n^{3-\varepsilon'})$ for $(\min, +)$–**Verification** (for some $\varepsilon' > 0$).

Hint: Use known reductions.

Exercise 4 (*16 points*) Consider the following graph problem:

Negative Matching Walk: Given a weighted graph $G = (V, E, w)$, an alphabet Σ, vertex labels $l : V \to \Sigma$, and a string $s = s_1 s_2 \dots s_T \in \Sigma^T$, determine whether there is a walk $v_1, \dots, v_T \in V$ in G, such that

- the labels $l(v_1), l(v_2), \dots, l(v_T)$ form the string s, i.e.

$$l(v_1)\, l(v_2)\, \dots\, l(v_T) = s,$$

- the total edge weight of the walk is negative, i.e.

$$\sum_{1 \le i < T} w(v_i, v_{i+1}) < 0.$$

a) *(3 points)* Demonstrate a deterministic algorithm for **Negative Matching Walk** that runs in time $O(T \cdot n^2)$.

b) *(6 points)* Show the following fine-grained reduction for the case $T = n$:

$$(\mathbf{APSP}, n^3) \le_{fgr} (\mathbf{NegativeMatchingWalk}, n^3).$$

We will now proceed to show that **NSETH** implies that there are no tight **SETH**-based lower bounds for this problem.

c) *(1 point)* Demonstrate a nondeterministic algorithm for **Negative Matching Walk** that runs in time $O(n^{3-\varepsilon})$ for some $\varepsilon > 0$ and $T = n$.

d) *(6 points)* Demonstrate a co-nondeterministic algorithm for **Negative Matching Walk** that runs in time $O(n^{3-\varepsilon'})$ for some $\varepsilon' > 0$ and $T = n$.

Hint: Let $u_j^{(i)}$ denote the minimum weight of a walk ending in vertex j that forms the string $s_1 \dots s_i$. Guess $u^{(i)}$ for all i and verify them using one $(\min, +)$ product verification (and other suitable checks).