Multivariate Algorithmics:
Problem set 1

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Due: Before the lecture on October 23, 2018   Tutorial: October 25, 2018

Problem 1.1 (✐) Prove: A directed graph $D = (V, A)$ is acyclic if and only if there is an order $\prec$ on V such that for every arc $(u, v) \in A$, we have that $u \prec v$.

Problem 1.2 (✐) Consider the following problem: Given a set of n points in the plane, decide whether there is a set of k lines such that every point lies on at least one line. Prove that this problem has a kernel of size $O(k^2)$.

Problem 1.3 Given an undirected graph $G = (V, E)$, a subset of vertices $X \subseteq V$ is a *feedback vertex set* if G after removal of X and all edges incident to X is a forest. Show that the problem of deciding whether a graph has a feedback vertex set of size at most k has a kernel with $O(k)$ vertices on regular, undirected graphs. [1]

Problem 1.4 The problem Connected Vertex Cover is defined as follows: Given an undirected graph G and a positive integer k, decide whether there exists a vertex cover C of G of size at most k, and such that the subgraph of G induced by C is connected.

(a) Where do the kernelization rules for Vertex Cover used to prove Theorem 2.4 fail in the connected case?

(b) Show that the problems admits a kernel with at most $2^k + O(k^2)$ vertices.

(c) Show that if the input graph does not contain a 4-cycle as a subgraph, then the problem admits a kernel with at most $O(k^2)$ vertices.

Problem 1.5 (☠) Extend the argument of the previous exercise to show that, for every fixed $d \geq 2$, Connected Vertex Cover restricted to graphs that do not contain the biclique $K_{d,d}$ admits a kernel with $O(k^d)$ vertices.

Problem 1.6 (☠) Let $G = (V, E)$ be a graph, and let $(x_v)_{v \in V}$ be an (not necessarily half-integral) optimum solution to the linear programming formulation of the vertex cover problem, LPVC(G). Define $(y_v)_{v \in V}$ as follows:

$$y_v = \begin{cases} 0 & \text{if } x_v < 1/2 \\ 1/2 & \text{if } x_v = 1/2 \\ 1 & \text{if } x_v > 1/2. \end{cases}$$

Show that $(y_v)_{v \in V}$ is also an optimal solution to LPVC(G).

Please note:

- Each problem is worth one point. There are no half points.
- You need a third of all points from the problem sets to be admitted to the exam.
- You can hand in assignments in groups of size up to three.

[1] The degree of the regular graph is *not* assumed to be a fixed constant.