Multivariate Algorithmics:
Problem set 4

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Problem 4.1 In the *Partial Dominating Set* problem, we are given an undirected graph G and positive integers k and t, and the goal is to check whether there exists a set $X \subseteq V(G)$ of size at most k such that $|N_G[X]| \geq t$. Obtain an algorithm running in time $2^{O(t)} n^{O(1)}$ for the problem.

Problem 4.2 (✐) Let n be a positive integer and consider the vector space E_n which is defined to be the linear span (over $\mathbb{R}$) of the basis $\{e_S \mid S \subseteq \{1, \dots, n\}\}$. That is, vectors of E_n are of the form

$$\sum_{S \subseteq \{1,\dots,n\}} \lambda_S e_S \,,$$

where $\lambda_S \in \mathbb{R}$ for every S. We define a vector multiplication $\wedge$ over (the basis of) E_n as follows:

$$e_S \wedge e_T = \varphi(S,T) \cdot e_{S \cup T} \,,$$

where $\varphi(S,T) = 0$ if $S \cap T \neq \emptyset$ and $\varphi(S,T) = \mathsf{sgn}(\sigma_{S,T})$[1] otherwise. Here $\sigma_{S,T}$ is the permutation that, given the sequence of elements of S and T, each ordered, outputs the ordered sequence over all elements of $S \cup T$. **Example:** $\varphi(\{4,2\},\{2,3,1\}) = 0$ and $\varphi(\{5,4\},\{2,3,1\}) = \mathsf{sgn}(14253)$ because the permutation (14253) orders the sequence $4,5,1,2,3$, which again is the sequence of $\{5,4\}$ and $\{2,3,1\}$ where each set is ordered.

Prove or disprove: For every $e \in E_n \setminus \{0\}$ it holds that $e^2 = 0$.

Problem 4.3 (☠) In the *Closest String* problem, we are given a set of k strings $x_1, \dots, x_k$ over alphabet Σ, each of length L, and a positive integer d. The goal is to find a string y of length L such that the *Hamming Distance*[2] between y and x_i is bounded by d for every $i \in \{1, \dots, k\}$. Prove that the problem is fixed-parameter tractable when parameterized by k and $|\Sigma|$.
Hint: Formulate the problem as an ILP and use the fact that ILP is fixed-parameter tractable when parametereized by the number of variables.
Bonus Point: Show that the problem is fixed-parameter tractable even when parameterized only by k.

Problem 4.4 (✐) In the *Cycle Packing* problem, we are given an undirected graph G and a positive integer k, and the goal is to check whether there exist k cycles in G that are pairwise vertex disjoint. Prove that the problem is nonuniformly fixed-parameter tractable, when parameterized by k.

Problem 4.5 Prove that *Undirected Hamiltonicity*[3] is fixed-parameter tractable when parameterized pathwidth. You may assume the existence of an algorithm $\mathbb{A}$ that, given an undirected graph G and a positive integer k, computes a (nice) path decomposition of width k of G or correctly decides that there is no such path decomposition. Furthermore $\mathbb{A}$ runs in time $f(k)n^{O(1)}$. (It is not allowed to apply algorithmic meta-theorems such as Courcelle's theorem. Instead you are required to construct a dynamic programming algorithm over the path decomposition of the input graph.)

Problem 4.6 (☠) Show that the pathwidth of an n-vertex tree is at most $\lceil \log n \rceil$. Construct a class of trees of pathwidth k and $O(3^k)$ vertices.

[1] The sign of a permutation is $(-1)^m$ where m is the number of transpositions, e.g. $\mathsf{sgn}(14253) = \mathsf{sgn}((14)(42)(25)(53)) = (-1)^4 = 1$.
[2] The Hamming Distance between two strings x and y of the same length is the number of positions i such that $x_i \neq y_i$.
[3] That is, given an undirected graph G, decide whether G contains an Hamilton Cycle.