**Multivariate Algorithmics:**
**Problem set 6**

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Due: Before the lecture on January 08, 2019 Tutorial: January 17, 2019

**Problem 6.1** (✎) Show that the dependency on $k$ in the Excluded Grid Theorem needs to be $\Omega(k^2)$. That is, show a graph of treewidth $\Omega(k^2)$ that does not contain a $k \times k$ grid as a minor.

**Problem 6.2** (✎) Give an example of a graph $G$ and sets $A, B \subseteq V(G)$ for which the submodularity inequality

$$|\Delta(A)| + |\Delta(B)| - |\Delta(A \cap B)| \geq |\Delta(A \cup B)|$$

is sharp.

**Problem 6.3** Show that the following problems are bidimensional: Feedback Vertex Set, Induced Matching, Cycle Packing, Longest Path and Dominating Set.

**Problem 6.4** The algorithm for Hamiltonian Cycle presented in the lecture suffers a multiplicative $O(n \log^{O(1)} n)$ overhead in the running time because of performing arithmetic operations on $O(n \log n)$-bit numbers. Show that one can shave this overhead down to $O(\log n)$ at the cost of getting a one-sided error Monte Carlo algorithm with false negatives occurring with probability $O(1/n)$.

**Problem 6.5** Prove that a set function $f : 2^{V(G)} \to \mathbb{R}$ is submodular if and only if

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B) \qquad (*)$$

holds for every $A \subseteq B \subseteq V(G)$ and $v \in V(G)$. Informally, inequality $(*)$ says that the marginal value of $v$ with respect to the superset $B$ (that is, the increase of value if we extend $B$ with $v$) cannot be larger than with respect to a subset $A$.

**Problem 6.6** (☠) In Digraph Pair Cut, the input consists of a directed graph $G$, a designated vertex $s \in V(G)$, a family of pairs of vertices $\mathcal{F} \subseteq \binom{V(G)}{2}$ and an integer $k$. The goal is to find a set $X$ of at most $k$ edges of $G$, such that for each pair $\{u, v\} \in \mathcal{F}$, either $u$ or $v$ is not reachable from $s$ in the graph $G - X$. Show an algorithm solving Digraph Pair Cut in time $2^k \cdot n^{O(1)}$ for an $n$-vertex graph G.