Multivariate Algorithmics:
Problem set 7

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`http://bit.ly/MulAlg18`

Due: Before the lecture on January 22, 2019 Tutorial: January 24, 2019

Problem 7.1 Show that Edge Multiway Cut is polynomial-time solvable on trees.

Problem 7.2 (✐) Shortest Path can be solved in time $O(m \cdot \log n)$ by Dijkstra's Algorithm on connected graphs without negative edge-weights. It can be solved in time $O(m \cdot n)$ by the Bellmann-Ford-Algorithm on connected graphs that may contain negative edge-weights. Show that it can be solved in time $O(m \cdot \text{poly}(k, \log n))$ where k is the number of edges with negative weights. Recall that the algorithm must either compute the length of a shortest path from a given node s to a given node t or correctly report the existence of a negative-weight cycle.

Problem 7.3 (✐) Let Φ be a graph property expressible in monadic second-order logic. Show that the problem Verify$[\Phi]$ of deciding whether Φ holds on a given graph G is solvable in time $f(k) \cdot n$ for some computable function f where k is the size of the largest vertex cover of the input graph.

Problem 7.4 (☠, 2 Points) In this exercise we generalize the principle of Möbius inversion to finite partially ordered sets (posets) and apply it to graph homomorphisms. To this end, let P be a poset. The *incidence algebra* of a poset $(P, \leq)$ is defined as follows:

$$\mathbb{I}(P, \leq) := \{A \in \mathbb{C}^{P \times P} \mid x \not\leq y \Rightarrow A(x, y) = 0\}\,.$$

One example of an element of $\mathbb{I}(P, \leq)$ is the so-called *zeta function*:

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

(1 Point) Consider the element μ of $\mathbb{I}(P, \leq)$, which is called the *Möbius function* over $(P, \leq)$ and which is inductively defined as follows:

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \not\leq y \\ -\sum_{x \leq z < y} \mu(x, z) & \text{otherwise} \end{cases}$$

- Show that:

$$\sum_{x \leq z \leq y} \mu(x, z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

- Show that $\mu = \zeta^{-1}$ and conclude from $\zeta \cdot \mu = \text{id}$ that the following identity holds as well:

$$\sum_{x \leq z \leq y} \mu(z, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

- **Möbius inversion:** Let $f, g : P \to \mathbb{C}$ such that $g(x) = \sum_{y \leq x} f(y)$ for all $x \in P$. Prove that the following holds for all $x \in P$:

$$f(x) = \sum_{y \leq x} \mu(y, x) \cdot g(y)$$

(1 Point) Let H be a graph with vertices V. Given two partitions σ and ρ of V, we write $\sigma \to \rho$ if ρ can be obtained from σ by joining two blocks of σ. Consider for example $\sigma = \{\{1,4\},\{2\},\{3\}\}$ then $\sigma \to \{\{1,2,4\},\{3\}\}$. Now let $\leq$ be the reflexive-transitive closure of $\to$, i.e., $\sigma \leq \rho$ iff there are $\sigma_1, \ldots, \sigma_k$ such that $\sigma \to \sigma_1 \to \cdots \to \sigma_k \to \rho$. Note that k might be zero.

- Let $P(V)$ be the set of partitions of V. Show that $(P(V), \leq)$ is a poset. Find the minimum $\bot$ and the maximum $\top$ of the poset.
- Given an element $\sigma \in P(V)$, the graph H/σ is obtained from H by contracting every block of σ to a single vertex and deleting multiple edges and self-loops. Given a graph G, we let $\mathsf{Hom}(H,G)$ be the number of graph homomorphisms from H to G and let $\mathsf{Inj}(H,G)$ be the number of injective graph homomorphisms from H to G. Use Möbius inversion to prove that
$$\mathsf{Inj}(H,G) = \sum_{\sigma \in P(V)} \mu(\bot,\sigma) \cdot \mathsf{Hom}(H/\sigma, G) \tag{1}$$
- Given graphs H and G, it is known that the number of subgraphs of G that are isomorphic to H equals $\mathsf{Aut}^{-1}(H) \cdot \mathsf{Inj}(H,G)$, where $\mathsf{Aut}(H)$ is the number of bijective homomorphisms from H to H. Discuss the algorithmic consequences of (1) w.r.t. the counting version of Subgraph Isomorphism.

Problem 7.5 Prove or disprove: The Isolation Lemma (Lemma 11.5) still holds if one replaces the sum in the definition of **w** by a product, and the $\min$ by $\max$.