max planck institut informatik

SAARLAND UNIVERSITY
COMPUTER SCIENCE

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Exercises for Randomized and Approximation Algorithms

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter18/rand-apx-algo/

Exercise Sheet 9: The Primal-Dual Method I

To be handed in by **December 18th, 2018** via e-mail to André Nusser
(CC to Antonios Antoniadis and Marvin Künnemann)

Exercise 1 *(8 Points)* Consider the following linear program:

$$\begin{aligned} \text{minimize} \quad & -5x_1 + 8x_2 + 4x_3 \\ \text{subject to} \quad & x_1 + x_2 = 2 \\ & x_2 - x_3 \leq 3 \\ & 2x_1 - x_3 \geq -1 \\ & x_1 \geq 0 \\ & x_3 \leq 0 \end{aligned}$$

(i) Formulate the dual of the linear program. *(3 Points)*

(ii) Rewrite the (primal) linear programm, so that the constraints are in standard form ($Ax \geq b$). *(3 Points)*

Exercise 2 *(11 Points)* Consider the following maximization problem:

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \leq b, \\ & x \geq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

and its corresponding dual:

$$\begin{aligned} \text{minimize} \quad & b^T y \\ \text{subject to} \quad & A^T y \geq c, \\ & y \geq 0, \\ & y \in \mathbb{R}^m \end{aligned}$$

and consider some feasible solutions x^* and y^* for the respective problems. Assume that there exist $\lambda, \mu > 0$ so that the following approximate complementary slackness conditions hold:

$$x_i^* > 0 \Rightarrow \sum_{j=1}^{m} a_{ij} y_j^* \leq \mu c_i,$$

and

$$y_i^* > 0 \Rightarrow \sum_{i=1}^{n} a_{ij} x_i^* \geq \lambda b_j.$$

Prove that x^* is a $\frac{\lambda}{\mu}$-approximation for the primal.

Exercise 3 *(21 Points)* The *local ratio* technique is closely related to the primal dual method; however, it only uses duality in an implicit way. Consider the following local ratio algorithm for the set cover problem. We compute a collection I of indices of a set cover, where I is initially empty. In each iteration, we find some element e_i not covered by the current collection I. Let ϵ be the minimum weight of any set containing e_i. We subtract ϵ from the weight of each set containing e_i; some such set has now weight zero and we add its index to I.

Let ϵ_j be the value of ϵ in the j'th iteration of the algorithm.

(i) Show that the cost of the solution returned is at most $f \sum_j \epsilon_j$. *(3 Points)*

(ii) Show that the cost of the optimal solution must be at least $\sum_j \epsilon_j$. *(3 Points)*

(iii) Conclude that the algorithm is an f-approximation algorithm. *(3 Points)*

The local ratio technique depends upon the *local ratio theorem.* For a minimization problem Π with weights w, we say that a feasible solution F is *α-approximate with respect to w* if the weight of F is at most α times the weight of an optimal solution given weights w.

Theorem 1 (Local Ratio Theorem)**.** *If there are nonnegative weights w such that $w = w^1 + w^2$, where w^1 and w^2 are also nonnegative weights, and we have a feasible solution F such that F is α-approximate with respect to both w^1 and w^2, then F is α-approximate with respect to w.*

(iv) Prove the local ratio theorem. *(5 Points)*

(v) Explain how the set cover algorithm above can be analyzed in terms of the local ratio theorem to prove that it is an f-approximation algorithm. *(7 Points)*