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Exercises for Randomized and Approximation Algorithms

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter18/rand-apx-algo/

Exercise Sheet 10: The Primal-Dual Method II

To be handed in by **January 15th, 2019** via e-mail to André Nusser
(CC to Antonios Antoniadis and Marvin Künnemann)

Exercise 1 *(8 Points)* Prove that the shortest s-t-path algorithm that we saw in the lecture is equivalent to Dijkstra's algorithm: that is, in each step it adds the same edge as Dijkstra's algorithm would add.

Exercise 2 *(10 Points)* Consider the *multicut problem in trees*. In this problem, we are given a tree $T = (V, E)$, k pairs of vertices s_i-t_i, and costs $c_e \geq 0$ for each edge $e \in E$. The goal is to find a minimum-cost set of edges F such that for all i, s_i and t_i are in different connected components of $G' = (V, E - F)$.

Let P_i be the set of edges in the unique path in T between s_i and t_i. Then we can formulate the problem as the following integer program:

$$\begin{aligned} \text{minimize} \sum_{e \in E} c_e x_e & \\ \text{subject to} \sum_{e \in P_i} x_e \geq 1, & \qquad 1 \leq i \leq k, \\ x_e \in \{0, 1\}, & \qquad e \in E. \end{aligned}$$

Suppose we root the tree at an arbitrary vertex r. Let $depth(v)$ be the number of edges on the path from v to r. Let $lca(s_i, t_i)$ be the vertex v on the path from s_i to t_i whose depth is minimum. Suppose we use the primal dual method to solve this problem, where the dual variable we increase in each iteration corresponds to the violated constraint that maximizes $depth(lca(s_i, t_i))$.

Prove that this gives a 2-approximation algorithm for the multicut problem in trees.

Exercise 3 *(12 Points)* In the *k-path partition problem*, we are given a complete, undirected graph $G = (V, E)$ with edge costs $c_e \geq 0$ that obey the triangle inequality, and a parameter k such that $|V|$ is a multiple of k. The goal is to find a minimum-cost collection of paths of k vertices such that each vertex is on exactly one path.

In the $0 \pmod{k}$-*tree partition problem*, we are given the same input as in the k-path partition problem except that the graph is not necessarily complete and the edge costs do not necessarily obey the triangle inequality. The goal is to find a minimum-cost collection of trees such that each tree has $0 \pmod{k}$ many vertices, and each vertex is in exactly one tree.

(i) Given an α-approximation algorithm for the second problem, produce a $2\alpha(1-\frac{1}{k})$-approximation algorithm for the first. *(4 Points)*

(ii) Use the primal-dual method to give a 2-approximation algorithm for the second problem. *(4 Points)*

(iii) Give a $4\left(1-\frac{1}{k}\right)$-approximation algorithm for the problem of partitioning a graph into cycles of length exactly k. *(4 Points)*

Exercise 4 *(10 Points)* Show that the performance guarantee of the primal-dual algorithm (from the lecture) for the uncapacitated facility location algorithm can be strengthened in the following way. Suppose that the algorithm opens the set T' of facilities and constructs the dual solution (v, w). Show that

$$\sum_{j \in D} \min_{i \in T'} c_{ij} + 3 \sum_{i \in T'} f_i \leq 3 \sum_{j \in D} v_j.$$