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Exercises for Randomized and Approximation Algorithms

www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter18/rand-apx-algo/

Exercise Sheet 11: Derandomization

To be handed in by **January 22nd, 2019** via e-mail to André Nusser
(CC to Antonios Antoniadis and Marvin Künnemann)

Exercise 1 *(15 Points)* Consider the following two-player game. The game begins with k tokens placed at the number 0 on the integer number line spanning $[0, n]$. Each round, one player, called the chooser, selects two disjoint and nonempty sets of tokens A and B. (The sets A and B need not cover all the remaining tokens; they only need to be disjoint.) The second player, called the remover, chooses one of the sets: all the tokens from this set are taken off the board, and the tokens from the other set all move up one space on the number line from their current position. The chooser wins if any token ever reaches n. The remover wins if the chooser finishes with one token that has not reached n.

a) *(3 Points)* Give a winning strategy for the chooser when $k \geq 2^n$.

b) *(6 Points)* Use the probabilistic method to show that there must exist a winning strategy for the remover when $k < 2^n$.

c) *(6 Points)* Explain how to use the method of conditional expectations to derandomize the winning strategy for the remover when $k < 2^n$.

Exercise 2 *(7 Points)* Let $k \geq 2$ and consider the Max-k-Cut problem: Given an edge-weighted undirected graph $G = (V, E)$, find a k-*cut*, i.e., partition of the vertices into k parts, such as to maximize its weight, i.e., the total weight of edges crossing from one of the k sets to another. Adapt the approach of derandomizing via pairwise independence to give a deterministic approximation algorithm for Max-k-Cut.
Hint: You may assume existence of a 2-universal hashing family $\mathcal{H}$ from $[n]$ to $[m]$ for any $n \geq m$ with the following properties: (1) we can sample some $h \in \mathcal{H}$ by sampling $O(\log n)$ uniform random bits, and (2) we can evaluate any h in polynomial time (given as representation the $O(\log n)$ random bits).

Exercise 3 *(7 Points)* Take your favourite randomized rounding approximation algorithm for the Max-SAT problem developed in the lecture or exercises and derandomize it using the method of conditional expectation.

Exercise 4 *(11 Points)* You aim to conduct an experiment involving n participants. Each participant displays a subset of m features. Your aim is to divide the participants into two groups such that for each feature, both groups have roughly the same number of members displaying this feature.

Formally, you are given an $n \times m$ matrix $F_{i,j}$ over $\{0,1\}$, where $F_{i,j} = 1$ indicates that participant i displays feature j. The objective is to find a set $G_1 \subseteq [n]$ such as to minimize

$$\max_{j \in [m]} \left| \sum_{i \in G_1} F_{i,j} - \sum_{i \in G_2} F_{i,j} \right| ,$$

where $G_2 := [n] \setminus G_1$.

Give a polynomial-time algorithm that finds a solution of objective value $O(\sqrt{n \log m})$.

Hint: You may make use of the following variant of the Chernoff bound (without proof): Let $X_1, \ldots, X_n$ *be independent random variables with* $\Pr[X_i = -1] = \Pr[X_i = 1] = \frac{1}{2}$. *Let* $X = \sum_{i=1}^{n} X_i$. *Then for any* $a > 0$, *we have*

$$\Pr[|X| \geq a] \leq e^{-\frac{a^2}{2n}}.$$