

Exercise 6: Can't see the Tree for the Forests!

Task 1: Radical Deforestation

We are given a complete weighted graph $G = (V, \binom{V}{2}, W)$. The goal is to find an extremely fast variant of the GHS algorithm in this setting. W.l.o.g., we assume that the system is synchronous. Message size is $\mathcal{O}(\log n)$, where we assume that an edge weight fits into a message. W.l.o.g., all edge weights are distinct and node identifiers are $1, \dots, n$.

- a) Suppose that at the beginning of phase i , the set T_i of previously selected MST edges is known to all nodes. Denote by \mathfrak{C} the set of (node sets of) connectivity components of (V, T_i) . For $C, C' \in \mathfrak{C}$, denote by $e(C, C')$ the lightest edge between C and C' , and define $\delta := \min\{\min_{C \in \mathfrak{C}}\{|C|\}, |\mathfrak{C}| - 1\}$. Find a subroutine that permits to determine for each $C \in \mathfrak{C}$ the lightest δ edges $e(C, \cdot)$ in $\mathcal{O}(1)$ rounds! At the end, the nodes in C must know these edges.

Hint: For component C , use not only the edges within the component, but also those between C and all other nodes. This way you still don't overuse any edges! Exploit that $|\mathfrak{C}| \leq n$ and that only the lightest edge $e(C, C')$ between any pair of components C, C' is of interest.

- b) Use a) to devise an MST algorithm that requires $\mathcal{O}(1)$ rounds per phase, maintains that at the beginning of each phase, all nodes know all previously determined MST edges, and in each phase, each component C is merged with at least δ other components!
- c) Prove that the algorithm from b) computes the MST in $\mathcal{O}(\log \log n)$ rounds!

Hint: Bound the minimal size of a component after phase i from below!

- d)* Solve a) again with $\delta(C) := \min\{|C| + 1, |\mathfrak{C}| - 1\}$. Does this significantly improve the analysis from part c)?

Task 2: Steiner Trees

The *Steiner tree* problem is a generalization of the MST problem. The goal is to find a minimum cost tree connecting a set of *terminals* $T \subseteq V$. This problem is NP-hard to approximate within factor $96/95$; we'll settle for a 2-approximation.

Definition 1 (Terminal graph). *Given a connected simple weighted graph $G = (V, E, W_G)$ and terminals $T \subseteq V$, the terminal graph is $(T, \binom{T}{2}, W_T)$, where $W_T(s, t)$ is the minimum weight of a path from s to t in G . (For a path $p = (v_0, \dots, v_k)$, its weight is $\sum_{i=1}^k W_G(v_{i-1}, v_i)$.)*

Consider an optimal Steiner tree O .

- a) Show that when visiting all nodes of O and returning to the root in the order given by a depth-first-search using the tree edges, each edge of O is traversed exactly twice. This is also called an *Euler tour* of O .
- b) List the nodes of T according to the order in which they are visited in the Euler tour of O . Denote this list by $(t_1, \dots, t_{|T|})$. Show that

$$\sum_{i=1}^{|T|-1} W_T(t_i, t_{i+1}) \leq 2W_G(O).$$

(The weight of an edge set is the sum of the edge weights.)

- c) Conclude that an MST of the terminal graph induces a spanning subgraph¹ whose weight is at most twice the weight of O !
- d) Assuming an MST of the terminal graph is already constructed and a corresponding edge set is already known (in the sense that nodes know their incident edges in this set), can you construct a 2-approximate solution that is a tree fast? How fast is your post-processing routine?

This is only one fragment of a distributed algorithm that constructs a 2-approximate Steiner tree. But fear not! We will devise an efficient such algorithm in this manner in a future exercise.

Task 3*: The Traveling Salesman

In the *Traveling Salesman Problem (TSP)*, a set of cities V is connected by charter flights $E \subseteq \binom{V}{2}$ of cost $W(e)$. The goal is to visit each city exactly once, connecting them by a Hamiltonian cycle, while minimizing the total cost of chartered flights.

In this exercise, we consider a complete graph $G = (V, \binom{V}{2}, W)$. Furthermore, we say that G is a *metric* if

$$\forall v, w \in V: \quad \text{dist}(v, w) = W(v, w),$$

where $\text{dist}(\cdot, \cdot)$ is the weighted distance.

- a) Consider the following algorithm: (1) Determine an MST on G . (2) Traverse V in the order they appear in a depth-first-search on the MST. Show that this yields a 2-approximation of TSP if G is a metric.
- b) Show that the algorithm from a) does not yield a 2-approximation if G is not a metric.
- c) Research how the size of the largest TSP instances that could be solved/approximated at the time changed over the years. How much of this is to be attributed to hardware improvements and how much to algorithmic improvements?
- d) Tell about your journey in the exercise session!

¹In fact, it might have arbitrarily weird cycles in it!