If you have questions regarding the exercises, please ask them on the mailing list.
Please hand in your solutions by sending them to Johannes at jbund@mpi-inf.mpg.de or directly in the lecture. If possible form groups of 2-3 people.
Do *not* send them to the public mailing list. The deadline is listed on the website.
Exercises marked with a star are not mandatory.

Exercise 1: It's a Colorful Life

Task 1: How the Colors Get into the Trees

Before you start this task, you may want to look at the chapter "Notation and Preliminaries" again, especially the section "Trees and Forests".

a) Change the Cole-Vishkin algorithm from the lecture so that it requires only $1/2 \cdot \log^* n + \mathcal{O}(1)$ rounds. The result should still be a message passing algorithm, so don't use pointer jumping!

   Hint: Compare what information can be gathered locally in T rounds to what the Cole-Vishkin algorithm actually relies on.

b) Find a message passing algorithm that 3-colors a rooted tree, i.e., a tree in which each non-root node initially knows which neighbor is its parent, in $\log^* n + \mathcal{O}(1)$ rounds.

   Hint: Leverage the same observation as for part a).

c) Show how to $(\Delta + 1)$-color a graph of maximum degree $\Delta \in \mathcal{O}(1)$ in $\log^* n + \mathcal{O}(1)$ rounds.

   Hint: Decompose the graph into a collection of Δ-many rooted forests.

Task 2: Sorry, but they just all look alike to me!

Given a graph $G = (V, E)$, an *independent set* $I \subseteq V$ satisfies that there is no edge $e \in E$ so that $e \subseteq I$, i.e., I contains no pair of neighbors in the graph. A *maximal independent set (MIS)* is an independent set $I \subseteq V$ such that $I \cup \{v\}$ is not independent for any $v \in V \setminus I$. Note that this is more general than a maxim*um* independent set.

Suppose that in the doubly linked list, we want to join nodes to sublists of 2 or 3 nodes under control of the same processor. For simplicity we will assume that the list is large enough (at least 5 elements) and cyclic.

a) Show that an MIS algorithm (i.e., one that computes an MIS) can be used to construct such sublists using $\mathcal{O}(1)$ additional rounds!

b) Show that an algorithm computing such sublists can be used to compute a 3-coloring of the list in $\mathcal{O}(1)$ additional rounds!

c) Show that an algorithm computing a 3-coloring can be used to compute an MIS in $\mathcal{O}(1)$ additional rounds!

d) What can you infer about the time complexity of optimal algorithms for these tasks?

e)* Show that an algorithm computing an MIS on arbitrary graphs can be used to compute a $(\Delta + 1)$-coloring of a graph of maximum degree Δ!

Hint: Replace each node by a clique (a.k.a. complete graph) of $\Delta+1$ nodes. Interpret each clique node as one of the possible colors of the original node. Add edges so that no adjacent cliques (original nodes) will have the same color if you compute an MIS of the new graph. Let each node simulate its entire clique in the MIS algorithm.

Task 3*: Theory and practice

Astrophysics[1] gives us the estimate that the evolution of the observable universe until now is equivalent to a computation on at most 10^{90} bits[2]. We will use this vast number as an upper bound on n, the initial amount of colors.

a) Argue that 5 rounds of Cole-Vishkin are practically *always* sufficient.

b) Execute 2 rounds of Cole-Vishkin with the following initial colors, written in decimal, appearing in this order:

$$1, 1337, 10^{90}, 23, 42$$

Each processor has the predecessor on the left. So the processor with color 1337 has the predecessor 1, and the processor with color 1 has the predecessor 42.

Hint: How is the least significant bit and evenness related? If this exercise seems laborous, make sure you use little-endian, and not big-endian.
Also, note that 1337 = `0b10011100101000000000...` in little endian.

c) Christoph likes to call $\log^*$ a "slow-growing constant". Can you come up with an even slower-growing "constant" that is actually unbounded?

[1] a science known for its reasonably-sized numbers

[2] `https://arxiv.org/pdf/quant-ph/0110141.pdf`