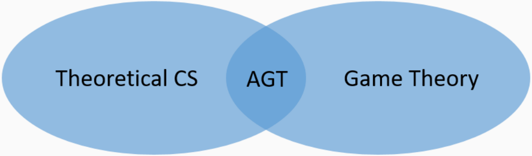
Topics in Algorithmic Game Theory and Economics

Game Theory from the Computer Scientist's point of view



Can we compute an "equilibrium" outcome of a game in polynomial time? (And more...)

Information

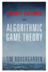
- Lectures: Wednesday, 14:15-16:00
- Homework: 4 or 5 homework sets
  - Half of points needed to qualify for exam.
- Exam: Oral examination, February 23-24, 2021
  - Covering lecture material and homework exercises.
- Tutorials: Doodle link given during break to check availability
- TA: Golnoosh Shahkarami

Material

- Books (for first part, until Christmas break):
  - Algorithmic Game Theory (Nisan, Roughgarden, Tardos, Vazirani)
  - Twenty Lectures on Algorithmic Game Theory (Roughgarden)

Nisan, Noam
Algorithmic game theory
Cambridge 2008
hide

- **print:** NIS n 2008:2 1.Ex
- **e-book, ip-range UdS**



Roughgarden, Tim
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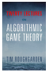

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- Some (elementary) background material for self-study:
  - Linear programming
  - Probability theory
  - Matroids

Tutorial "0" next week about background material.

Topics in Algorithmic Game Theory and Economics

Pieter Kleer

Max Planck Institute for Informatics (D1)
Saarland Informatics Campus

November 11, 2020

Lecture 1
Introduction and Overview

What is game theory?

Study of mathematical models of strategic interaction among (rational) players that influence each other's outcome.

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Study of mathematical models of strategic interaction among (rational) players that influence each other's outcome.

- Road users in traffic networks.
- Selfish routing of internet traffic.
- Online selling platforms.
- Auctions.

Two examples

Traffic networks

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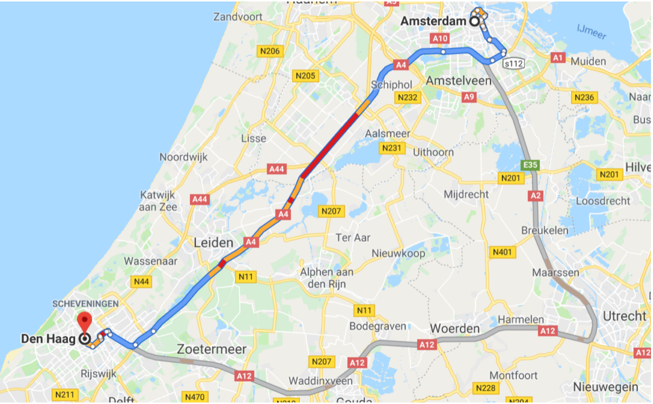

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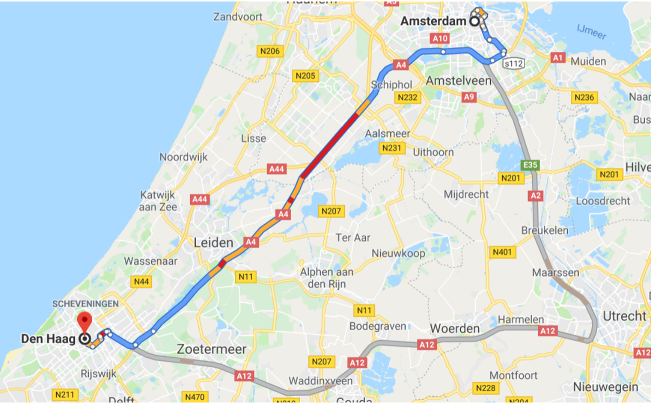

- Outcome is a driver's travel time from work to home.

Traffic networks (cont'd)

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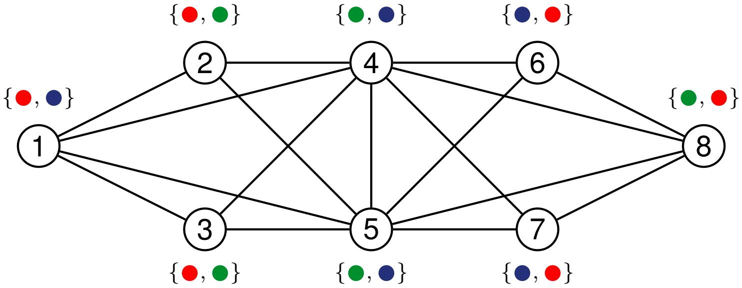
Conflicting interests:

- Road users want to get home as quickly as possible.
  - Goal: Minimize individual travel time.
- Government wants road network to be used efficiently.
  - Goal: Minimize total travel time in the network

Coordination games



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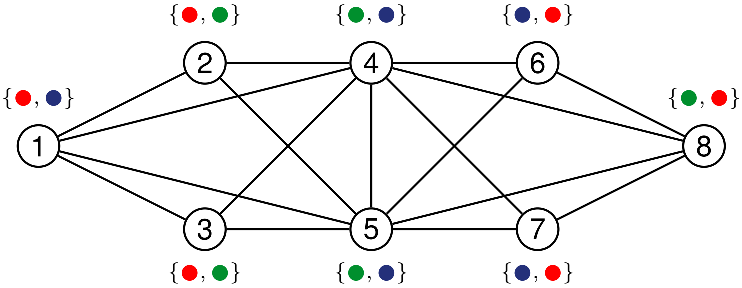

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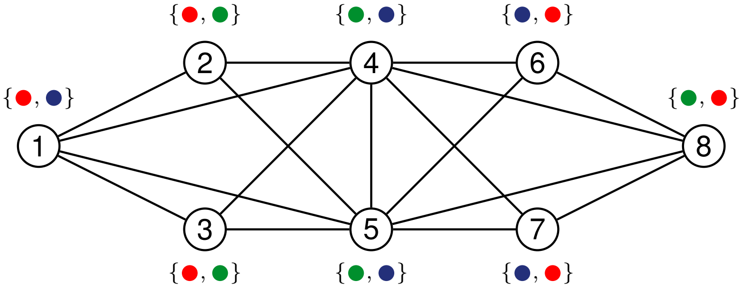

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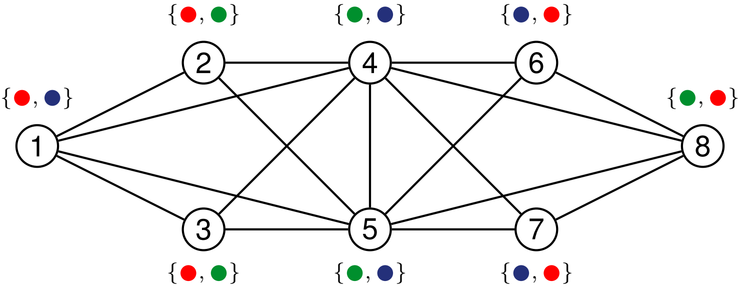

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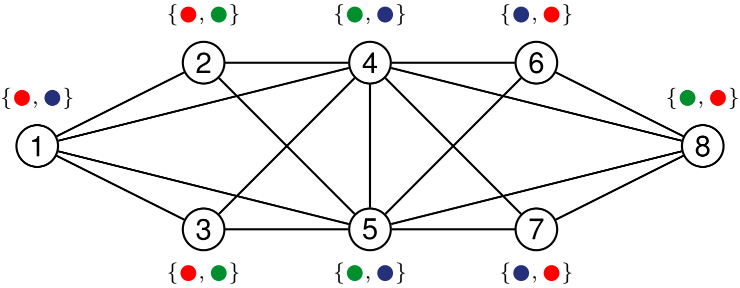

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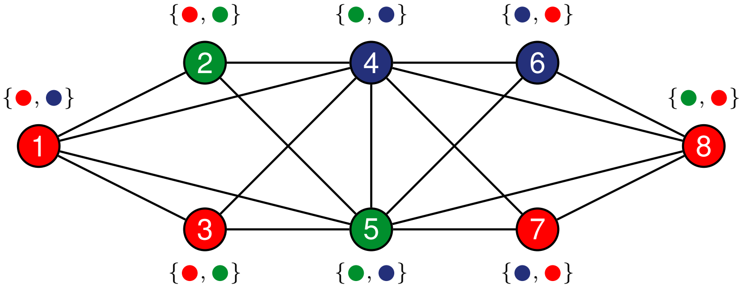

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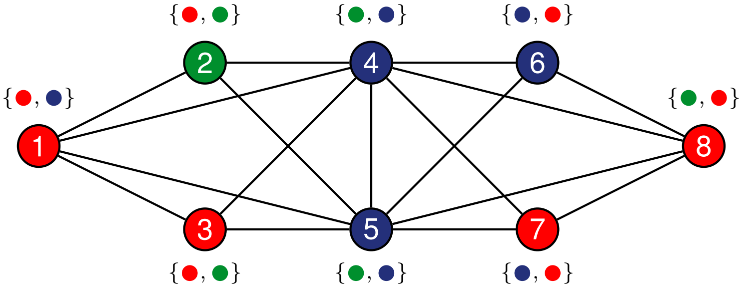

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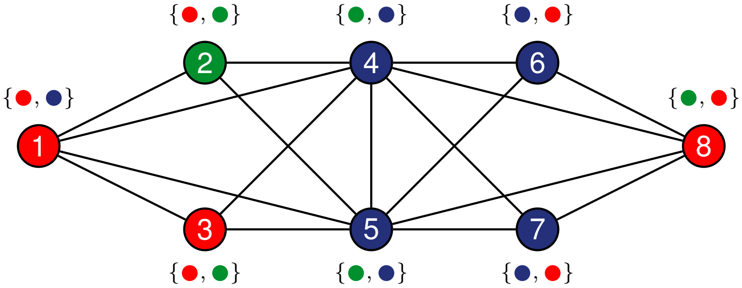

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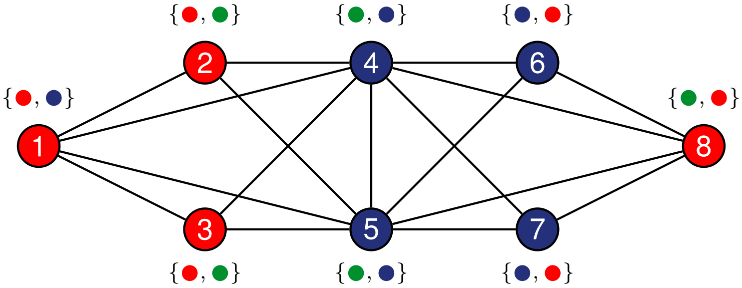

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- Equilibrium computation
  - Can we compute equilibrium in polynomial time?
- Inefficiency of equilibria
  - How much worse can $C(s)$ be compared to $C(s^*)$?

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  - Price of Anarchy (PoA)/Price of Stability (PoS).

Games and equilibrium concepts

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(All players have full information.)

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Some solution/equilibrium concepts:

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Famous thought experiment.

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Alice and Bob committed a crime. Police wants a confession.

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		Silent	*Betray*
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  - Similar game where communication is possible.

Dominant strategies

Definition (Dominant strategy)

A strategy $t_i \in \mathcal{S}_i$ is dominant for player $i \in N$ if

$$C_i(s_1, \ldots, t_i, \ldots, s_n) \leq C_i(s_1, \ldots, t_i', \ldots, s_n)$$

for every $t_i' \in \mathcal{S}_i$ and any strategy vector

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- Does not always exist.

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PNE not guaranteed to exist in general games.

- Existence is known for special class of congestion games.
  - Next lectures.

Matching pennies

PNE is not guaranteed to exist, already in very simple games.

Matching pennies

Alice and Bob both choose side of a penny.

		Bob	
		Head	*Tails*
Alice	*Head*	$(0, 1)$	$(1, 0)$
	Tails	$(1, 0)$	$(0, 1)$

- Alice wants both coins to be on the same side.
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Yes, randomize over both strategies.

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A mixed strategy $\sigma_i : \mathcal{S}_i \to [0, 1]$ of player $i \in N$ is a probability distribution over pure strategies in $\mathcal{S}_i$, i.e.,

$$\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \ \forall t \in \mathcal{S}_i \quad \text{and} \quad \sum_{t \in \mathcal{S}_i} \tau(t) = 1 \right\}.$$

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A collection of mixed strategies $\sigma = (\sigma_i)_{i \in N}$, with $\sigma_i \in \Delta_i$, is a mixed Nash equilibrium if

$$\mathbb{E}_{x \sim \sigma} [C_i(x)] \leq \mathbb{E}_{(x_i', x_{-i}) \sim (\sigma_i', \sigma_{-i})} \left[C_i(x_i', x_{-i}) \right] \quad \forall \sigma_i' \in \Delta_i. \tag{1}$$

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Theorem (Nash's theorem, 1950)

Any finite game Γ *has a mixed Nash equilibrium.*

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Is there an equilibrium concept that always exists and is computable?

Game of Chicken

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Alice and Bob both approach an intersection.

		Bob	
		Stop	*Go*
Alice	*Stop*	$(0, 0)$	$(3, -1)$
	Go	$(-1, 3)$	$(4, 4)$

- Two PNEs: (Stop, Go), (Go, Stop)
- One MNE: Both players randomize over Stop and Go.

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Distributions over strategy profiles (a, b) for these equilibria are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

- Sensible 'equilibrium' would be the strategy profile distribution

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Conditioned on this recommendation, the best thing for a player to do is to follow it.

Correlated equilibrium

Definition (Correlated equilibrium (CE))

A distribution σ on $\times_i \mathcal{S}_i$ is a correlated equilibrium if for every $i \in N$ and $x_i \in \mathcal{S}_i$, and every unilateral deviation $x_i' \in \mathcal{S}_i$, it holds that

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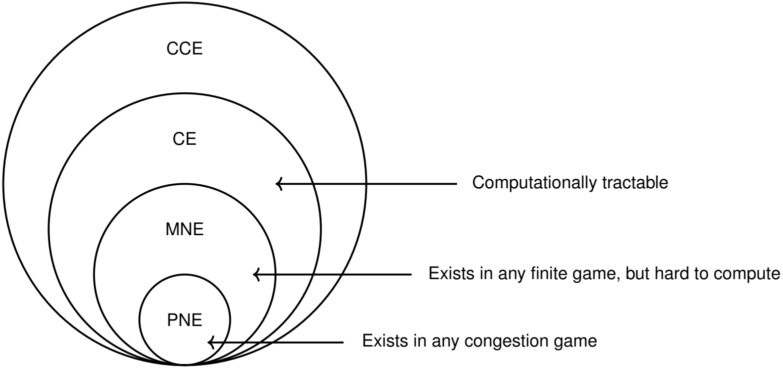
Hierarchy of equilibrium concepts

The concepts we have seen so far all are subsets of each other.



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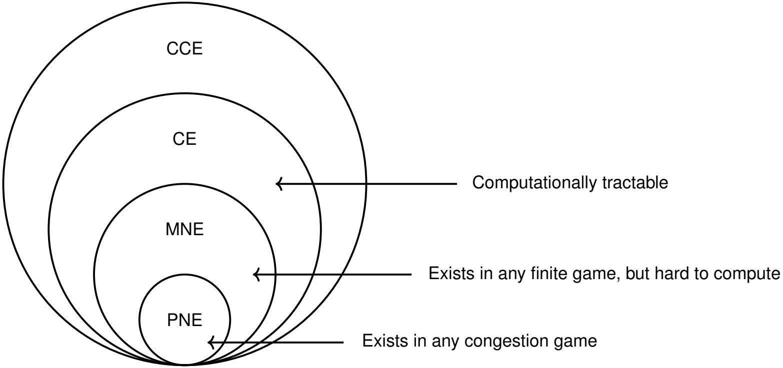

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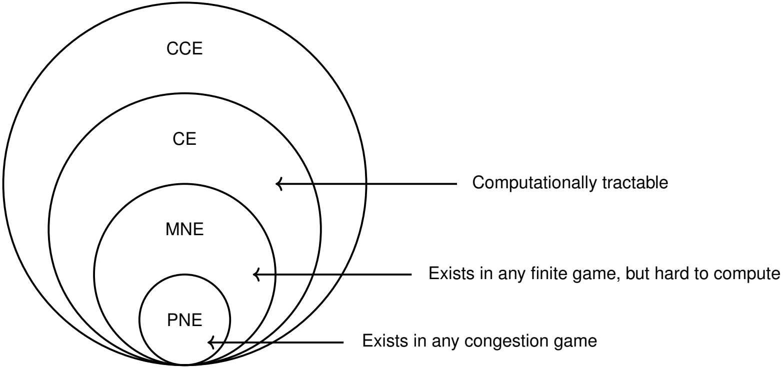

- Exercise: Prove that this is indeed a hierarchy.
  - Every PNE is an MNE, every MNE is a CE, etc.

Rough outline until Christmas

Congestion and potential games

- Existence of PNE.
- Computational complexity.
  - Complexity of computing PNE.
  - Complexity of best response dynamics.
- Inefficiency of equilibria.
  - Price of Anarchy/Stability.

General 2-player and n-player games

- Existence of MNE (Nash's theorem).
  - Discussion on computational complexity.
- Computation of approximate mixed Nash equilibria.
- Computation of (coarse) correlated equilibria.
  - Linear programming approach.
  - Decentralized dynamics.
- Inefficiency of MNE/CE/CCE.
  - Roughgarden's smoothness framework.

Background (prerequisite) material

Some tools from combinatorics, probability theory and optimization

Linear programming

Optimize linear function over set of linear constraints, e.g.,

$$\begin{array}{ll} \text{max} & x_1 + 3x_2 \\ \text{subject to} & x_1 + x_2 \leq 5 \\ & 3x_1 + x_2 \leq 2 \\ & x_1, x_2 \geq 0 \\ & x_1, x_2 \in \mathbb{R}. \end{array}$$

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- Might have seen this in, e.g., course "Optimization".

Probability theory

Basic knowledge about probability theory is assumed, in particular, we sometimes use concentration inequalities.

- Markov's inequality
- Chebyshev's inequality
- Chernoff bounds

Matroids

Generalization of linear independence of vectors in, e.g., $\mathbb{R}^n$.

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- No $v_i \in X$ can be written as linear combination of other vectors.

Example

$$E = \{v_1, v_2, v_3, v_2\} = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 17 \\ 34 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix} \right\}$$

Is $X = \{v_1, v_2, v_3\}$ independent?

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- Maximal independent sets are bases (of $\mathbb{R}^n$).

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- *Downward-closed*: $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$,
- *Augmentation property*:

$$A, C \in \mathcal{I} \text{ and } |C| > |A| \Rightarrow \exists e \in C \setminus A \text{ such that } A \cup \{e\} \in \mathcal{I}.$$

Sets in $\mathcal{I}$ are called independent sets.

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Sets in $\mathcal{I}$ are called independent sets.

Linear matroid: Let $E = \{v_i : i = 1, \ldots, k\} \subseteq \mathbb{R}^n$ and take

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Let E be finite set of elements (think of, e.g., vectors).

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which gives a contradiction.

Graphic matroid: Let $G = (V, E)$ be undirected graph and consider matroid $\mathcal{M} = (E, \mathcal{I})$, with ground the edges E of G, given by

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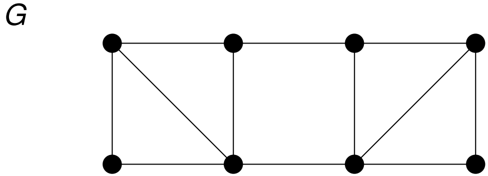

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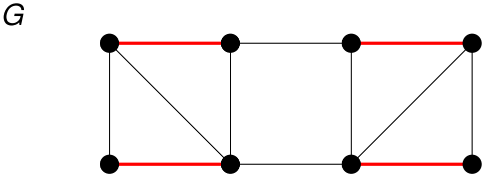

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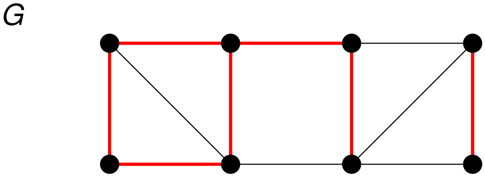

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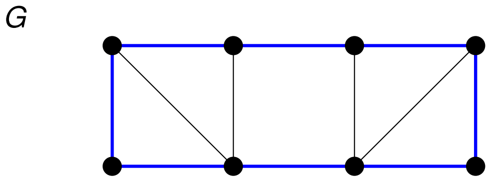

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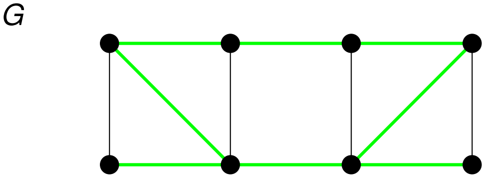

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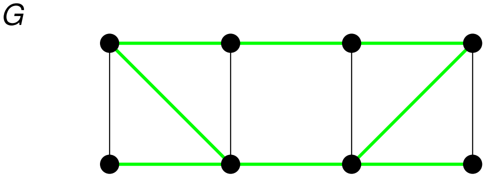

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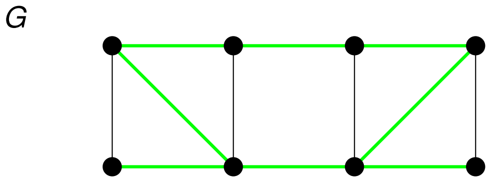

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