Topics in Algorithmic Game Theory and Economics

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Saarland Informatics Campus

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Lecture 10
Matroid Secretary Problems

Matroids (recap)

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$$E = \{v_1, v_2, v_3, v_2\} = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 17 \\ 34 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix} \right\}$$

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- Maximal independent sets are bases (of $\mathbb{R}^n$).

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Let $G = (V, E)$ be undirected graph and consider matroid $\mathcal{M} = (E, \mathcal{I})$, with ground the edges E of G, given by

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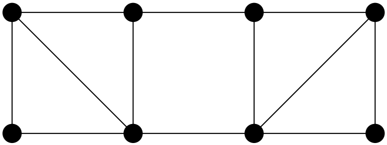

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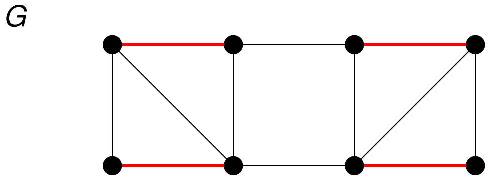

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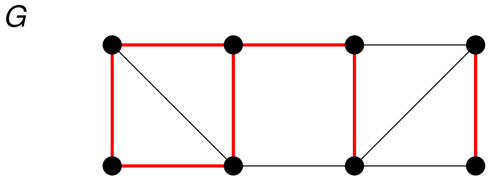

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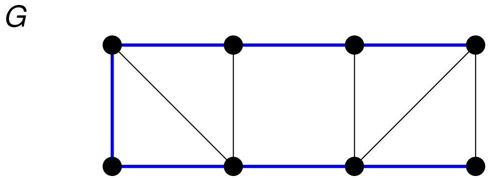

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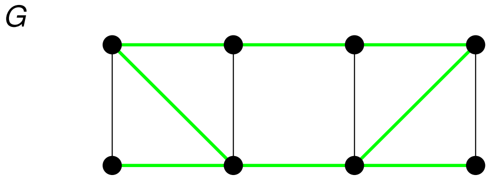

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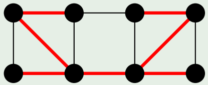

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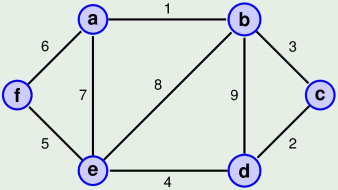

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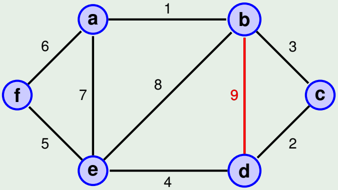

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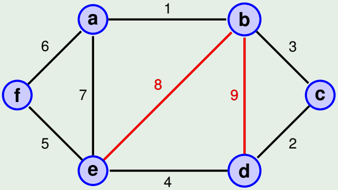

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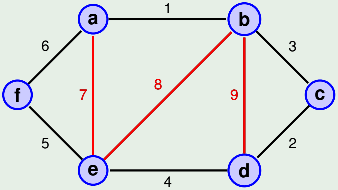

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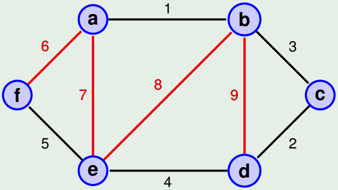

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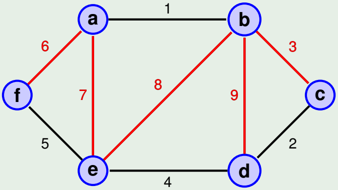

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- Upon arrival of $e \in E$, its weight $w_e \geq 0$ is revealed.

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- Would yield (another) generalization of secretary problem.

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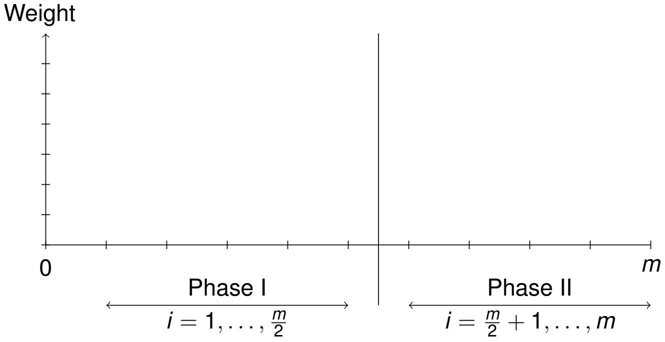
$$t = \frac{w}{2^j}.$$
- For $i = \frac{m}{2} + 1, \ldots, m$: Select $\sigma(i)$ if $w_{\sigma(i)} \geq t$ and $X + \sigma(i) \in \mathcal{I}$.



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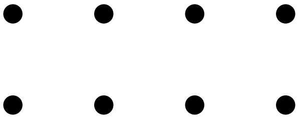

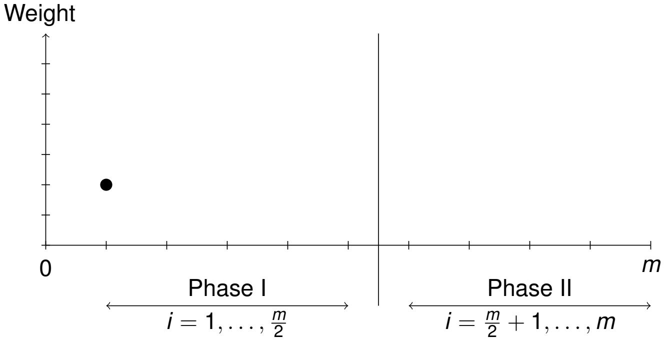

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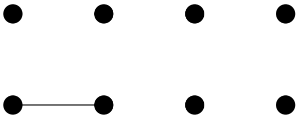

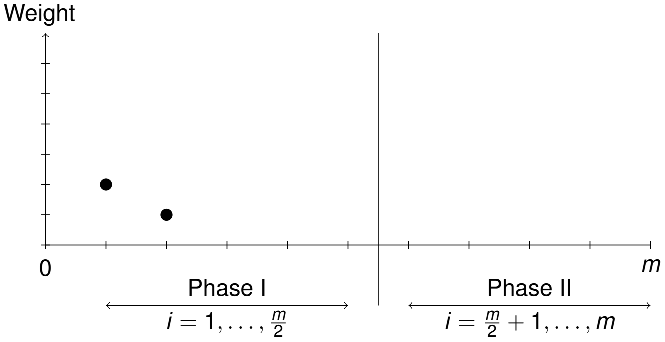

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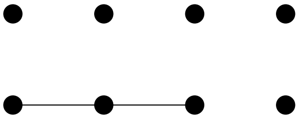

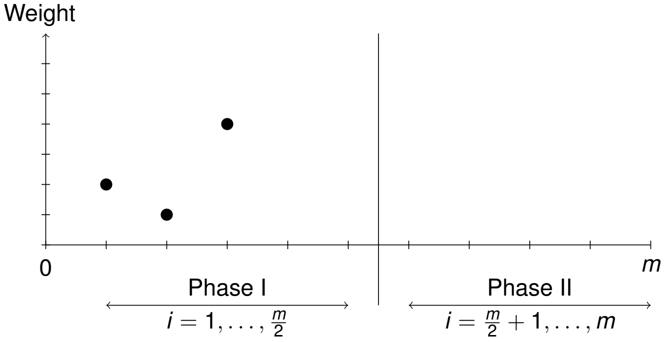

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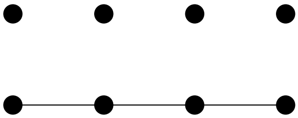

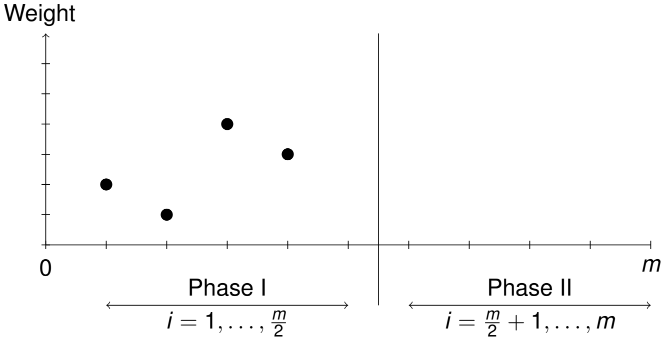

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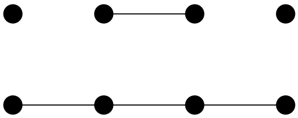

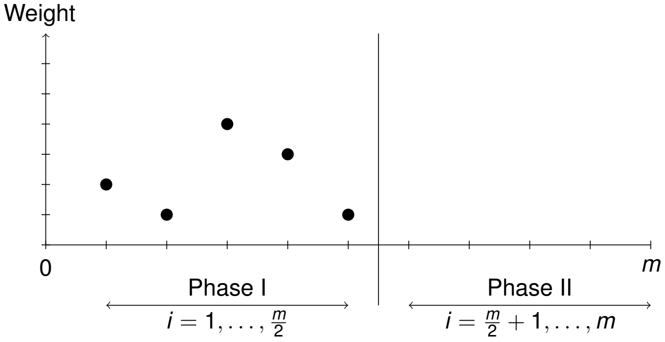

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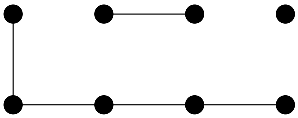

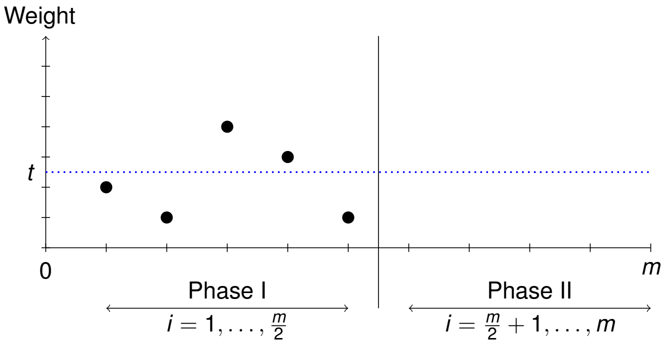

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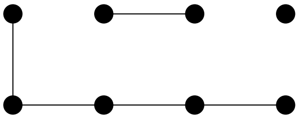

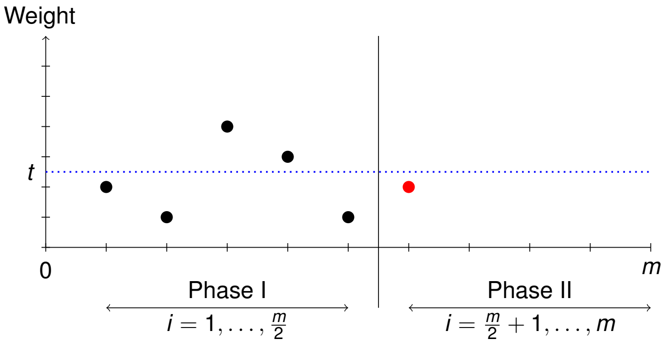

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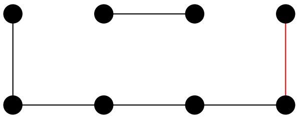

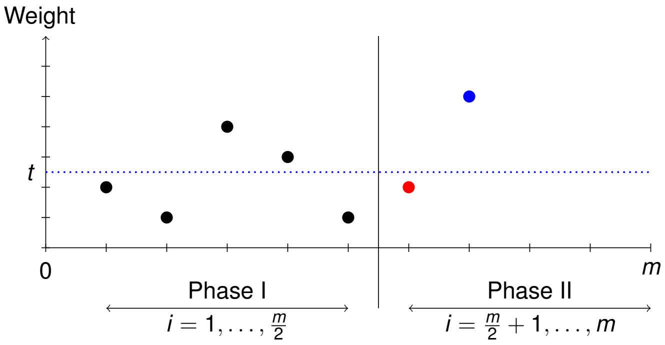

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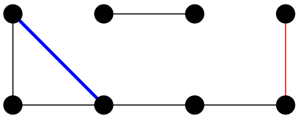

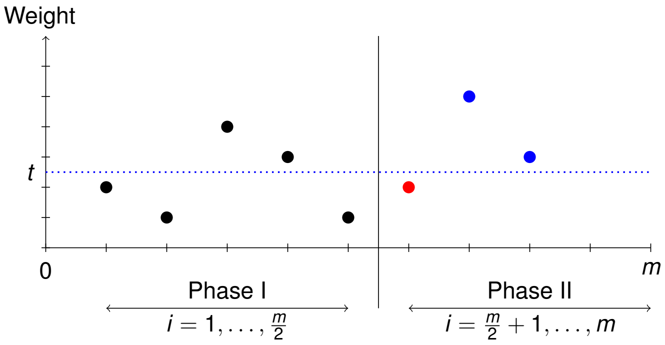

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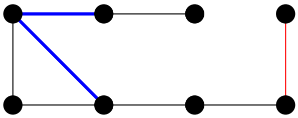

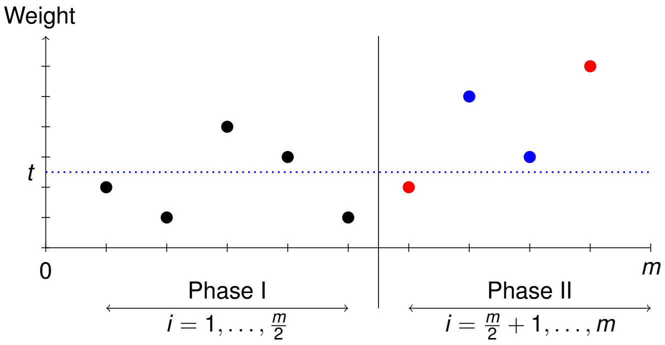

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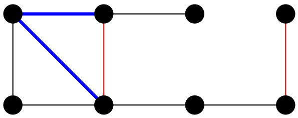

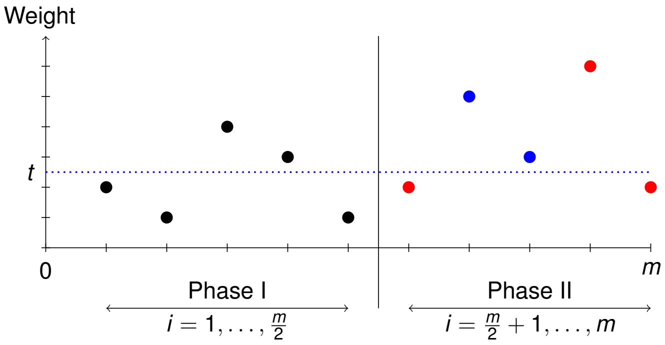

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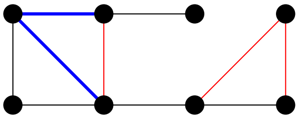

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To conclude,

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The random threshold algorithm is $\frac{1}{32(\lceil \log(r) \rceil + 1)}$*-approximation, where* r *is the rank of the matroid* $\mathcal{M} = (E, \mathcal{I})$*.*

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  - Elements are bidders that each can receive one "unit of stuff".
  - Matroid constraint on which combination of bidders can be allocated a unit.

Beyond matroids

Online selection problems

Consider

- Finite set of elements $E = \{e_1, \ldots, e_m\}$.
- Weight function $w : E \to \mathbb{R}_{\geq 0}$.
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Goal: Select (online) independent set $X \in \mathcal{F}$ of max. weight.

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Goal: Select (online) independent set $X \in \mathcal{F}$ of max. weight.

In general, for arbitrary downward-closed set systems, no constant-factor approximation exists.

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Theorem (Babaioff et al. (2007))

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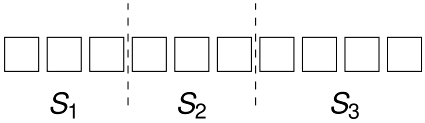

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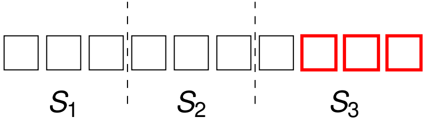

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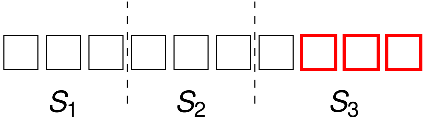

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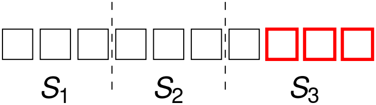

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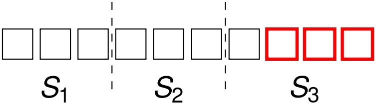

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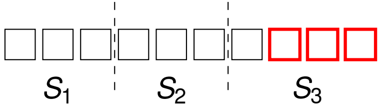

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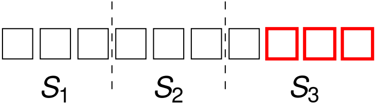

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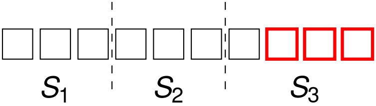

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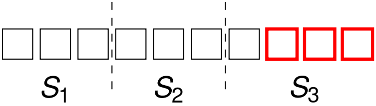

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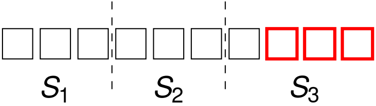

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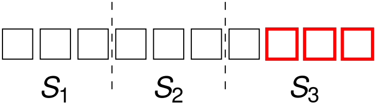

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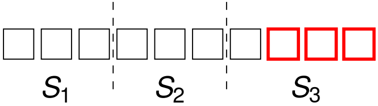

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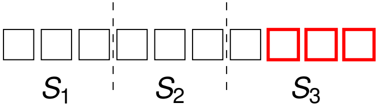
- Therefore, set selected by $\mathcal{A}$ has weight at most 2 in expectation.



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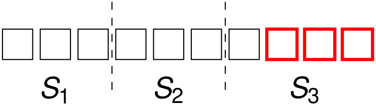

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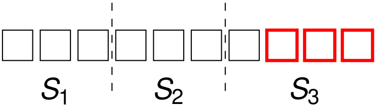

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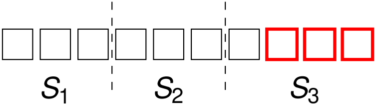

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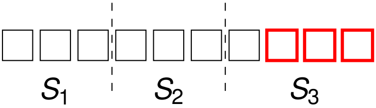

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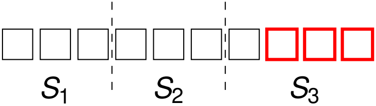

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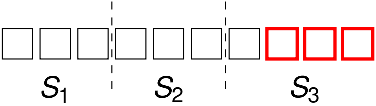

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- This is then tight up to a factor $\log\log(n)$.

Graphic matroid

Korula-Pál algorithm

Graphic matroid secretary problem

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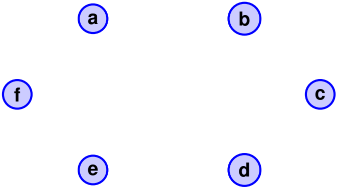

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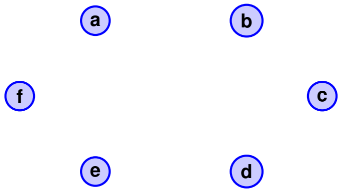

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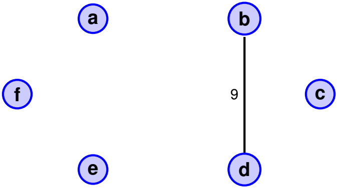

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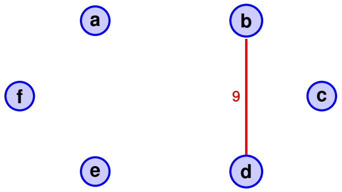

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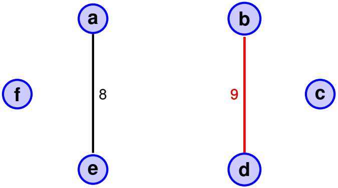

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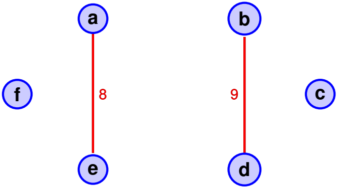

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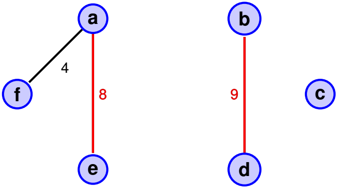

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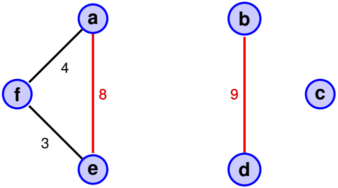

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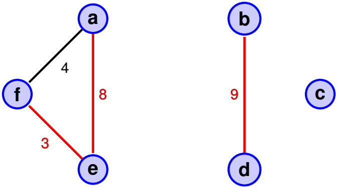
Online selection (with initially $X = \emptyset$)

- Edges of (known) graph $G = (V, E)$ arrive in unknown uniform random arrival order σ.
- Upon arrival of $e \in E$, its weight $w_e \geq 0$ is revealed.
- Decide irrevocably whether to accept or reject it.
  - Acceptance is only allowed if $X + e$ is forest of G.
  - That is, $X + e$ does not contain a cycle.



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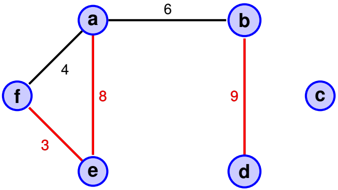

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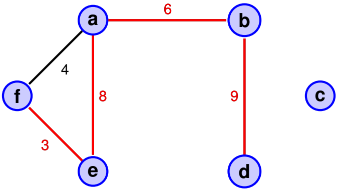

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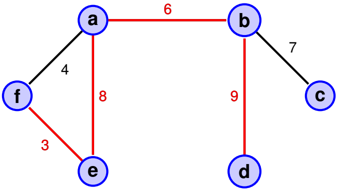

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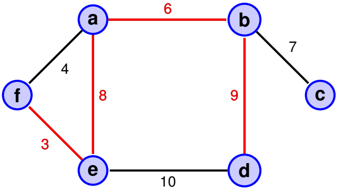

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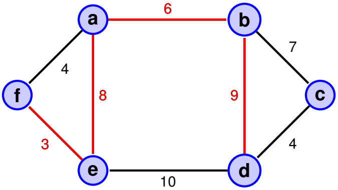

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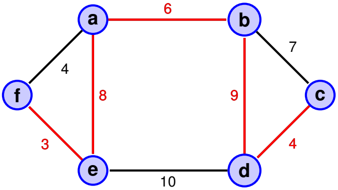

$1/(2e)$-approximation

Assume that $V = \{1, \ldots, n\}$.

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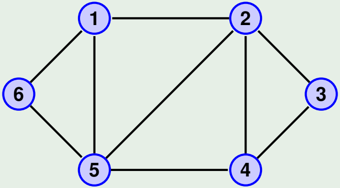
- For every $z \in V$ at most one arc from every A_z is selected.

Example (Every edge oriented to lowest index node)



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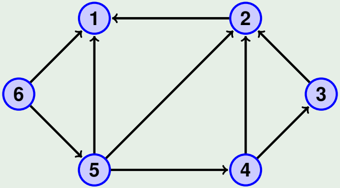

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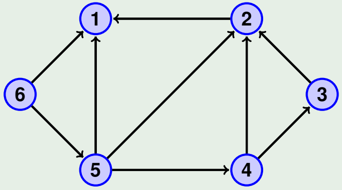

$A_1 = \{(6,1),(5,1),(2,1)\}$
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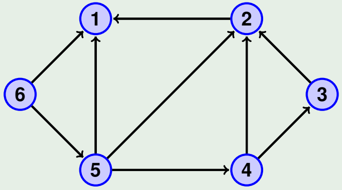

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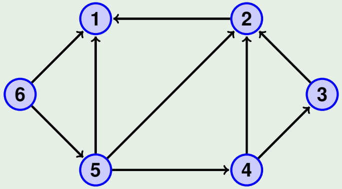

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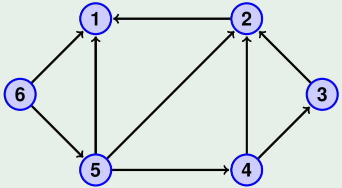

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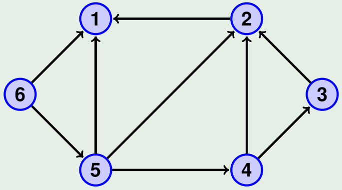

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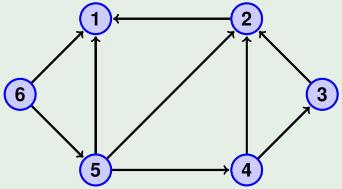

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Is there $\frac{1}{e}$-approximation for graphic matroid secretary problem?