Topics in Algorithmic Game Theory and Economics

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Lecture 10
Matroid Secretary Problems

Matroids (recap)

Matroids

Generalization of linear independence of vectors in, e.g., $\mathbb{R}^n$.

Let $E = \{v_1, \ldots, v_k\}$ be collection of vectors $v_i \in \mathbb{R}^n$ for all i.

- Assume that $k > n$ and $\text{span}(E) = \mathbb{R}^n$.

Subset of vectors $X \subseteq E$ is called linearly independent if, for $\gamma_i \in \mathbb{R}$,

$$\sum_{v_i \in X} \gamma_i \cdot v_i = 0 \quad \Rightarrow \quad \gamma_i = 0 \ \forall i.$$

- No $v_i \in X$ can be written as linear combination of other vectors.

Example

$$E = \{v_1, v_2, v_3, v_2\} = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 17 \\ 34 \end{pmatrix}, \begin{pmatrix} -4 \\ -2 \end{pmatrix} \right\}$$

Is $X = \{v_1, v_2, v_3\}$ independent? NO, because $v_3 = 3v_1 + 4v_2$.

- Maximal independent sets are bases (of $\mathbb{R}^n$).

Matroid

Definition (Matroid)

Set system $\mathcal{M} = (E, \mathcal{I})$ with non-empty $\mathcal{I} \subseteq 2^E = \{X : X \subseteq E\}$ is matroid if it satisfies the following:

- *Downward-closed*: $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$,
- *Augmentation property*:

$$A, C \in \mathcal{I} \text{ and } |C| > |A| \Rightarrow \exists e \in C \setminus A \text{ such that } A \cup \{e\} \in \mathcal{I}.$$

Sets in $\mathcal{I}$ are called independent sets.

Example (Linear matroid)

Let $E = \{v_i : i = 1, \ldots, k\} \subseteq \mathbb{R}^n$ and take

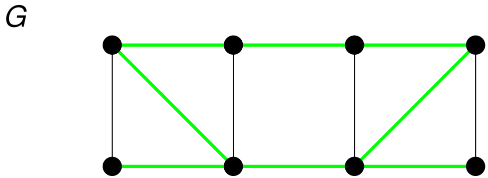
$$W \in \mathcal{I} \quad \Leftrightarrow \quad \text{vectors in } W \text{ are linearly independent.}$$

- *Augmentation property:* Note that if $|C| \geq |A| + 1$ and every $v_i \in C$ is a linear combination of vectors in A, then $\text{span}(C) \subseteq \text{span}(A)$, and hence $|C| = \dim(\text{span}(C)) \leq \dim(\text{span}(A)) = |A|$, which gives a contradiction.

Example (Graphic matroid)

Let $G = (V, E)$ be undirected graph and consider matroid $\mathcal{M} = (E, \mathcal{I})$, with ground the edges E of G, given by

$$W \in \mathcal{I} \quad \Leftrightarrow \quad \text{subgraph with edges of } W \text{ has no cycle.}$$



Bases of a matroid

Maximal independents set of a matroid $\mathcal{M} = (E, \mathcal{I})$ are called bases.

Definition (Base)

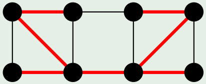
An independent set $X \in \mathcal{I}$ is a base if for every $e \in E \setminus X$ it holds that $X + e \notin \mathcal{I}$, i.e., no element can be added to X while preserving independence.

Lemma

All bases of a given matroid $\mathcal{M}$ have the same cardinality. This common cardinality r is called the rank of the matroid.

Example

- Bases of graphic matroid on $G = (V, E)$, with $|V| = n$, are spanning trees (when G is connected). Rank is $n - 1$.



(Offline) maximum weight independent set

Consider matroid $\mathcal{M} = (E, \mathcal{I})$ with $E = \{e_1, \ldots, e_m\}$.

- Rename elements such that $w_1 \geq w_2 \geq \cdots \geq w_m \geq 0$.

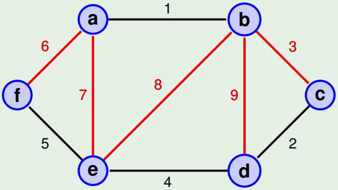
Greedy algorithm

Set $X = \emptyset$. For $i = 1, \ldots, m$:

- If $X + e_i \in \mathcal{I}$, then set $X \leftarrow X + e_i$.

In other words, greedily add elements while preserving independence.

Example (Graphic matroid)



Matroid secretary problem

Matroid secretary problem

Selecting maximum weight independent set online.

Given is matroid $\mathcal{M} = (E, \mathcal{I})$. Set $X = \emptyset$.

- Elements in E arrive in unknown uniform random arrival order σ.
- Upon arrival of $e \in E$, its weight $w_e \geq 0$ is revealed.
- Decide irrevocably whether to accept or reject it.
  - Acceptance is only allowed if $X + e$ is independent, i.e., $X + e \in \mathcal{I}$.

Matroid secretary problem: Select (online) independent set $X \in \mathcal{I}$ of maximum weight.

- In the offline setting, X is maximum weight base of the matroid.
- Generalization of the secretary problem.
  - Corresponds to the so-called 1-uniform matroid.
  - In k-uniform matroid, $X \in \mathcal{I}$ if and only if $|X| \leq k$.

Some literature

About the matroid secretary problem:

- Problem introduced by Babaioff, Immorlica and Kleinberg (2007).
  - They gave $\Omega\left(\frac{1}{\log(r)}\right)$-approximation.
  - Remember that r is rank of the matroid.
- State of the art: $\Omega\left(\frac{1}{\log\log(r)}\right)$-approximation.
  - First by Lachish (2014).
  - Simpler algorithm by Feldman, Svensson and Zenklusen (2015).
- Constant factor approximations known for various special cases
  - Graphic matroids, k-uniform matroids, laminar matroids, transversal matroids, and more.

> **Open question:** Does there exist, for an arbitrary matroid, a constant factor approximation?

- Stronger question: Does there exist a $\frac{1}{e}$-approximation?
- Would yield (another) generalization of secretary problem.

Matroid secretary problem

$\Omega\left(\frac{1}{\log(r)}\right)$*-approximation*

Random threshold algorithm

Consider (given) matroid $\mathcal{M} = (E, \mathcal{I})$ of rank r with $|E| = m$.

Random threshold algorithm for arrival order σ

Set $X = \emptyset$.

Phase I (Observation).

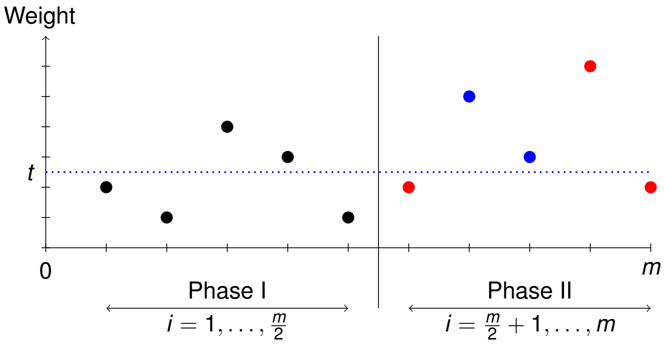
- For $i = 1, \ldots, \frac{m}{2}$: Reject $\sigma(i)$.

Phase II (Selection).

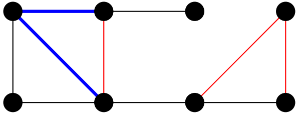
- Let $w = \max_{i=1,\ldots,m/2} w_{\sigma(i)}$, and choose $j \in \{0, 1 \ldots, \lceil \log(r) \rceil\}$ uniformly at random.
- Set threshold

$$t = \frac{w}{2^j}.$$

- For $i = \frac{m}{2} + 1, \ldots, m$: Select $\sigma(i)$ if $w_{\sigma(i)} \geq t$ and $X + \sigma(i) \in \mathcal{I}$.



Consider graphic matroid as example:



Analysis (sketch)

Theorem

The random threshold algorithm is a $\frac{1}{32(\lceil \log(r) \rceil + 1)}$-approximation, where r is the rank of the matroid.

Proof: Consider an optimal base $B^* = \{x_1, \ldots, x_r\}$.

- Assume that $w(x_1) > w(x_2) > \cdots > w(x_r)$.
- Let $1 \leq q \leq r$ be the largest number for which $w(x_q) \geq w(x_1)/r$.

> Let $w = (35, 14, 8, 6, 3, 2, 1)$, so that $r = 7$. Then $\frac{w(x_1)}{r} = 5$ and $q = 4$.

- Then it holds that

$$\sum_{i=1}^{q} w(x_i) \geq \frac{1}{2} \cdot w(B^*).$$

- Why?

$$\sum_{i=q+1}^{r} w(x_i) \leq \sum_{i=q+1}^{r} \frac{w(x_1)}{r} \leq w(x_1).$$

Remember we may focus on q largest elements in optimal base $B^* = \{x_1, \ldots, x_r\}$ with $w(x_1) \geq \cdots \geq w(x_q) \geq \cdots \geq w(x_r)$.

Some notation for (random) set T:

- Let $n_i(T)$ be the number of elements whose weight is at least $w(x_i)$.
  - Note that $n_i(B^*) = i$.
- Let $m_i(T)$ be the number of elements whose weight is at least $w(x_i)/2$.

Lemma

Let X be the set outputted by the random threshold algorithm. For $i = 1, \ldots, q$, we have (remember $n_i(B^*) = i$)

$$\mathbb{E}_\sigma[m_i(X)] \geq \frac{1}{8(\lceil \log(r) \rceil + 1)} \cdot i.$$

We first show how lemma leads to desired approximation guarantee.

$$\mathbb{E}_\sigma[m_i(X)] \geq \frac{1}{8(\lceil \log(r) \rceil + 1)} \cdot i.$$

Remember $m_i(X)$ is number of elements with weight at least $w(x_i)/2$ in X.

First note that (remember $n_i(B^*) = i$)

$$\begin{aligned} \sum_{i=1}^{q} w(x_i) &= \left[\sum_{i=1}^{q-1} (w(x_i) - w(x_{i+1})) n_i(B^*)\right] + w(x_q) n_q(B^*) \\ w(X) &\geq \frac{1}{2}\left[\sum_{i=1}^{q-1} (w(x_i) - w(x_{i+1})) m_i(X)\right] + \frac{1}{2} w(x_q) m_q(X) \end{aligned}$$

The approximation guarantee then follows as

$$\begin{aligned} \mathbb{E}_\sigma[w(X)] &\geq \frac{1}{2}\left[\sum_{i=1}^{q-1} (w(x_i) - w(x_{i+1})) \mathbb{E}_\sigma[m_i(X)]\right] + \frac{1}{2} w(x_q) \mathbb{E}_\sigma[m_q(X)] \\ &\geq \frac{1}{16(\lceil \log(r) \rceil + 1)} \left(\left[\sum_{i=1}^{q-1} (w(x_i) - w(x_{i+1})) i\right] + w(x_q) q\right) \\ &= \frac{1}{16(\lceil \log(r) \rceil + 1)} \sum_{i=1}^{q} w(x_i) \geq \frac{1}{32(\lceil \log(r) \rceil + 1)} w(B^*). \end{aligned}$$

□

Lemma

Let X be set outputted by algorithm. For $i = 1, \dots, q$,

$$\mathbb{E}_\sigma[m_i(X)] \geq \frac{1}{8(\lceil \log(r) \rceil + 1)} \cdot i$$

with $m_i(X)$ number of elements selected with weight at least $w(x_i)/2$.

Proof: Fix i and let A be the event that (both)

- The max. weight element x_1 in B^* appears in Phase I, and
- The chosen $j \in \{0, \dots, \lceil \log(r) \rceil\}$ has the property that

$$w(x_i) \geq t := \frac{w(x_1)}{2^j} \geq \frac{w(x_i)}{2}. \tag{1}$$

- (In the example, it could also be that $w(x_1)/r \geq w(x_i)/2$.)

Choice of q guarantees $w(x_i) \geq w(x_1)/r$, so at least one j satisfies (1):

$$\mathbb{P}(A) \geq \frac{1}{2(\lceil \log(r) \rceil + 1)}.$$

With probability $\mathbb{P}(A) \geq 1/(2(\lceil \log(r) \rceil + 1))$, chosen j is such that

$$w(x_i) \geq t = \frac{w(x_1)}{2^j} \geq \frac{w(x_i)}{2}.$$

For any $1 \leq j < i$, it holds that $w(x_j) \geq w(x_i) \geq t$.

- Every such x_j can potentially be chosen in Phase II as it exceeds the threshold.
  - It might be rejected still based on the independence criterium.

For given ordering σ, let Y be cardinality of maximal size independent set of threshold-exceeding elements that appear in Phase II.

- Because the set $\{x_2, \ldots, x_i\}$ is independent, it follows that

  $$\mathbb{E}_\sigma[Y \mid A] \geq \frac{i-1}{2} \geq \frac{i}{4}$$

  as every x_j appears in Phase II with prob. $1/2$.
- Here we use the fact that we are considering a matroid!

$$\mathbb{E}_\sigma[Y \mid A] \geq \frac{i-1}{2} \geq \frac{i}{4}$$

One might interpret Phase II as just greedily selecting elements that exceed the threshold t.

- Greedy algorithm (with weights equal to 1 for every element) implies that the size of the set chosen is at least Y.
- (Might also argue directly through the augmentation property.)

To conclude,

$$\mathbb{E}_\sigma[m_i(X)] = \mathbb{E}_\sigma[Y \mid A] \cdot \mathbb{P}(A) \geq \frac{1}{8(\lceil \log(r) \rceil + 1)} \cdot i.$$

□

Comments

Theorem

The random threshold algorithm is $\frac{1}{32(\lceil \log(r) \rceil + 1)}$*-approximation, where* r *is the rank of the matroid* $\mathcal{M} = (E, \mathcal{I})$*.*

- Algorithm can be adjusted to the setting where the rank of the matroid is unknown.
    - This makes analysis more complicated.
- “Single-threshold” algorithms can never give constant-factor approximation.
    - As shown by Babaioff et al. (2018).
- Problem can be turned into a randomized strategyproof mechanism.
    - Elements are bidders that each can receive one "unit of stuff".
    - Matroid constraint on which combination of bidders can be allocated a unit.

Beyond matroids

Online selection problems

Consider

- Finite set of elements $E = \{e_1, \ldots, e_m\}$.
- Weight function $w : E \to \mathbb{R}_{\geq 0}$.
- Downward-closed collection $\mathcal{F} \subseteq 2^E = \{X : X \subseteq E\}$.
  - Matroid set system (possibly) without augmentation property.

Online selection:

- Elements in E arrive in unknown uniform random arrival order σ.
- Upon arrival of $e \in E$, its weight $w_e \geq 0$ is revealed.
- Decide irrevocably whether to accept or reject it.
  - Acceptance is only allowed if $X + e \in \mathcal{F}$.

Goal: Select (online) independent set $X \in \mathcal{F}$ of max. weight.

In general, for arbitrary downward-closed set systems, no constant-factor approximation exists.

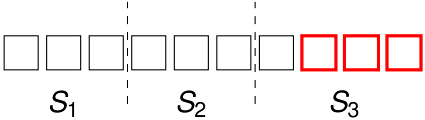
Online selection for general systems

Theorem (Babaioff et al. (2007))

There is no randomized algorithm that, for every downward-closed set system $\mathcal{F} = (E, \mathcal{I})$ with m elements and (random) weights in $\{0, 1\}$, obtains an approximation guarantee better than $O(\ln\ln(n)/\ln(n))$.

Proof (very informal): Let $n \geq 0$ be an integer and set $r = \ln(n)$.

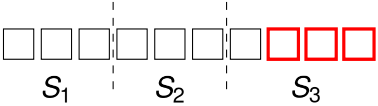
- $E = S_1 \cup S_2 \cup \cdots \cup S_k$ is disjoint union of sets S_i with $k = \lceil \frac{n}{r} \rceil$.
- Every S_i either has r or $r - 1$ elements.



- $X \subseteq E$ independent (i.e., $X \in \mathcal{F}$) $\Leftrightarrow$ $X \subseteq S_i$ for some $i = 1, \ldots, k$.

This set system is (structurally) very "far away" from a matroid.

> $X \subseteq E$ independent (i.e., $X \in \mathcal{F}$) $\Leftrightarrow X \subseteq S_i$ for some $i = 1, \ldots, k$.



$|S_i| \in \{r, r-1\}$

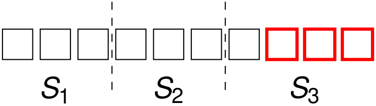
The weights are generated independently for every $e \in E$:

$$w_e = \begin{cases} 1 & \text{with probability } \frac{1}{r} \\ 0 & \text{with probability } 1 - \frac{1}{r} \end{cases}.$$

No (randomized) algorithm $\mathcal{A}$ can give constant-factor approximation.

What can we achieve online (sketch):

- As soon as $\mathcal{A}$ selects an element $e \in S_{i^*}$ (for some i^*), it can only pick subsequent elements from the same S_{i^*}.
- Elements from S_{i^*} that have not yet arrive, have total expected weight at most 1. (By definition of weights.)
- Therefore, set selected by $\mathcal{A}$ has weight at most 2 in expectation.



$|S_i| \in \{r, r-1\}$

What can we achieve offline (sketch):

- Balls-in-bins calculation shows that, in expectation, there will be always at least one S_i that has $\Omega(\ln(n)/\ln\ln(n))$ elements with weight 1.
- Offline optimum OPT $= \Omega(\ln(n)/\ln\ln(n))$ in expectation.

□

Final remark:

Theorem (Rubinstein, 2016)

There exists an $\Omega(1/\log(n))$-approximation w.r.t. the offline optimum for general downward-closed set system with weights in $\{0, 1\}$.

- This is then tight up to a factor $\log\log(n)$.

Graphic matroid

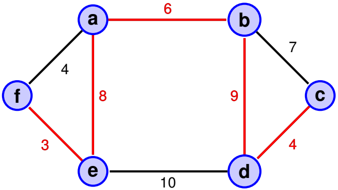
Korula-Pál algorithm

Graphic matroid secretary problem

For many special matroids, there exists a constant-factor approximation (often based on a reduction to secretary problems).

Online selection (with initially $X = \emptyset$)

- Edges of (known) graph $G = (V, E)$ arrive in unknown uniform random arrival order σ.
- Upon arrival of $e \in E$, its weight $w_e \geq 0$ is revealed.
- Decide irrevocably whether to accept or reject it.
  - Acceptance is only allowed if $X + e$ is forest of G.
  - That is, $X + e$ does not contain a cycle.



$1/(2e)$-approximation

Assume that $V = \{1, \dots, n\}$.

Graphic matroid secretary algorithm for graph $G = (V, E)$

Before the edges arrive:

- With prob. $\frac{1}{2}$ replace every edge $\{i, j\}$ ($i < j$) with arc (i, j), or
- with prob. $\frac{1}{2}$ replace every edge $\{i, j\}$ ($i < j$) with arc (j, i).

Let A be the resulting (random) set of directed arcs, and

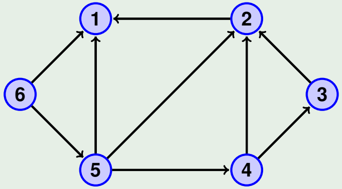
$$A_z = \{(u, z) \in A : \{u, z\} \in E\} \text{ for } z \in V.$$

When the edges arrive:

- Run (in parallel) the secretary algorithm on every A_z.

- We either orient every edge to its node with highest index, or every edge to its node with lowest index.
- A_z is set of all arcs that are oriented into z.
- For every $z \in V$ at most one arc from every A_z is selected.

Example (Every edge oriented to lowest index node)



$A_1 = \{(6,1),(5,1),(2,1)\}$
$A_2 = \{(5,2),(4,2),(3,2)\}$
$A_3 = \{(4,3)\}$
$A_4 = \{(5,4)\}$
$A_5 = \{(6,5)\}$
$A_6 = \emptyset$

Preprocessing.

- Randomly orient every edge to highest index, or every edge to lowest index.
- Resulting arcs A are partitioned into sets A_z for $z \in V$.

Running secretary algorithms on the A_z. For all $z \in V$ (in parallel):

- Phase I: First observe $\lfloor \frac{|A_z|}{e} \rfloor$ of edges contained in A_z.
- Phase II: Select first edge whose weight exceeds best weight seen in Phase I.

Graphic matroid secretary algorithm for graph $G = (V, E)$

Before the edges arrive:

- With prob. $\frac{1}{2}$ replace every edge $\{i, j\}$ $(i < j)$ with arc (i, j), or
- with prob. $\frac{1}{2}$ replace every edge $\{i, j\}$ $(i < j)$ with arc (j, i).

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When the edges arrive:

- Run (in parallel) the secretary algorithm on every A_z.

High-level steps to show it is $\frac{1}{2e}$-approximation:

- First show that indeed forest is outputted.
  - That is, an independent set of the graphic matroid.
- Then compare to (oriented) offline max. weight spanning tree.
- Give bound on expected contribution per node:
  - Factor $\frac{1}{2}$ is result of (randomly) orienting edges.
  - Factor $\frac{1}{e}$ is result of running (parallel) secretary algorithms.

Final remarks

By now, $\frac{1}{4}$-approximation for graphic matroid secretary problem is known.

- See paper of Soto, Turkieltaub and Verdugo (2018).
- Proof uses similar algorithm and analysis as that of Kesselheim et al. (2013) for online bipartite matching.
- Technique also applies to other special cases of matroids.

> Is there $\frac{1}{e}$-approximation for graphic matroid secretary problem?