Topics in Algorithmic Game Theory and Economics

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Saarland Informatics Campus

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Lecture 11
Prophet Inequalities

Online selection

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Goal: *Select subset $S \in \mathcal{F}$ maximizing $w(S) = \sum_{e \in S} w(e)$.*

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With adversarial arrival order

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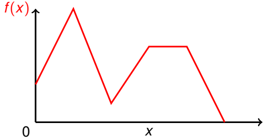

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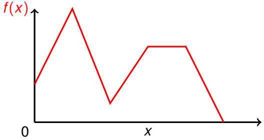

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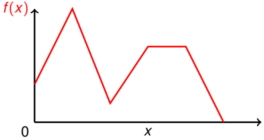

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Remark

All the results we discuss today hold for both continuous and discrete distributions, but sometimes need slightly different arguments.

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Adversary is non-adaptive if order is fixed after seeing all realizations.

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  - We don't know realization w_1, when deciding on element e_2.

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Prophet Inequality with $\alpha = \frac{1}{2}$

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Median of distribution X is value m such that

$$\mathbb{P}(X < m) \leq \frac{1}{2} \quad \text{and} \quad \mathbb{P}(X > m) \leq \frac{1}{2}.$$

- For continuous distributions, the median is the "middle value" of the distribution.

Selecting single item

Krengel, Sucheston and Garling (1978) show there is a prophet inequality with $\alpha = \frac{1}{2}$.

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Example

Suppose we have uniform distribution over (continuous) interval $[a, b]$. Then $m = \frac{a+b}{2}$.

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This completes the proof. □

Some remarks

Theorem (Kleinberg and Weinberg, 2012)

The KW-algorithm selects an element e^* with the property that

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Matroid prophet inequality

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Selecting indep. set from matroid $\mathcal{M} = (E, \mathcal{I})$ with arrival order σ.

Set $S = \emptyset$.

- For $i = 1, \dots, m$, a realization $w_i \sim X_i$ is generated.
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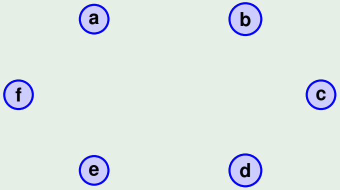

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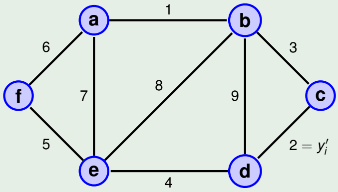

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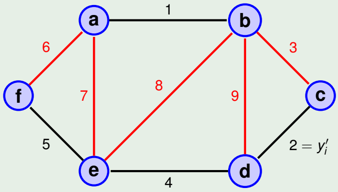

Base B'

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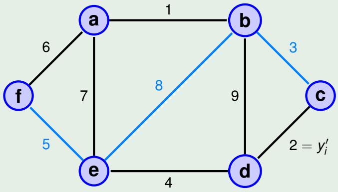

S selected so far

KW-algorithm for matroid constraint

Algorithm sets threshold in step i based on marginal contribution of $\sigma(i)$.

- Let $y' = (y'_1, \ldots, y'_m) \geq 0$ be given weights, and let B' be a max. weight base under y'.
- For given independent set $S \in \mathcal{I}$, we can augment S with elements $R(S) \subseteq B'$ so that $S \cup R(S)$ is base of $\mathcal{M}$.
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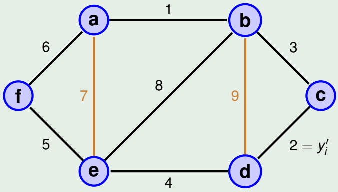

Augmentation $R(S)$

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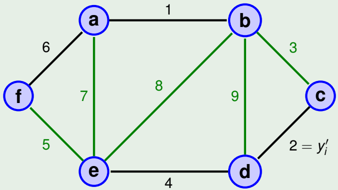

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Theorem (Babaioff et al. (2007), Rubinstein (2016))

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Selecting single element

Sample-based threshold

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- Algorithms only using single sample from every X_i will be called single-sample algorithms.

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Corollary (AKW, 2014)

There is a single-sample $\alpha = \frac{1}{8}$ graphic matroid prophet inequality.

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- Proof uses the fact that both (offline) sample x_i and (online) realization w_i come from the same distribution X_i.

Prophet inequalities for I.I.D. distributions

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The algorithm has access to the weights revealed so far, and the common distribution X.

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In case the online algorithm only has access to weights revealed so far (but not to common distribution X), there is a prophet inequality with $\alpha = \frac{1}{e}$ and this is best possible.

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Theorem (Correa, Saona and Ziliotto, 2019)

There is a secretary prophet inequality with $\alpha = 1 - \frac{1}{e} + \frac{1}{27} \approx 0.669$ for selecting a single element.

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- Also saw some other models (e.g., single-sample settings).