Topics in Algorithmic Game Theory and Economics

Pieter Kleer

Max Planck Institute for Informatics (D1)
Saarland Informatics Campus

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Lecture 2
Congestion Games I - Computation of PNE

Introduction

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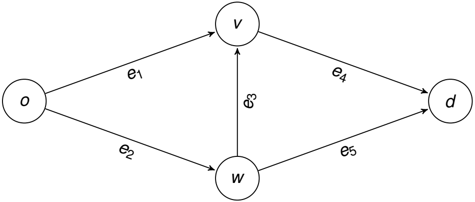
Studied extensively in the last twenty years in the area of AGT.

Atomic selfish routing game (example)

Given is directed graph $G = (V, E)$ with origin o and destination d.



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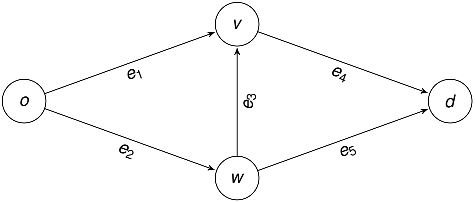

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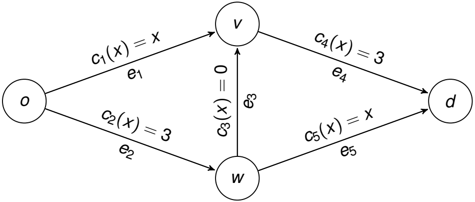

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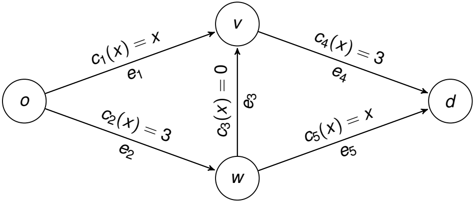

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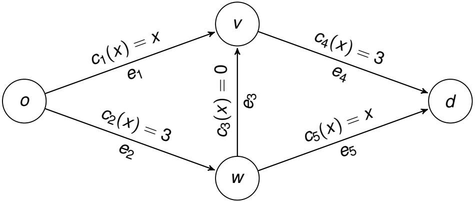

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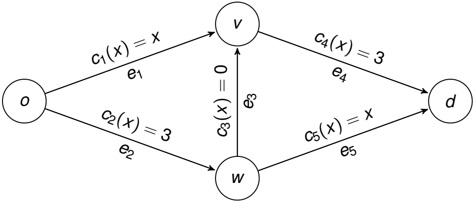

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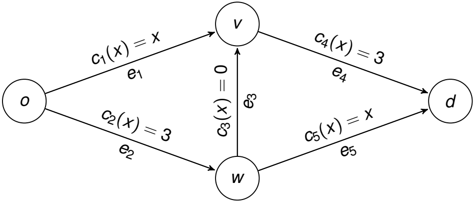

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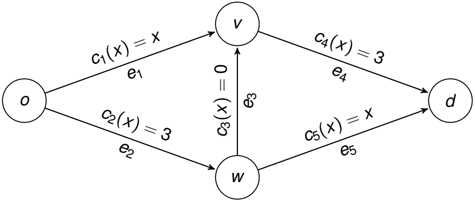

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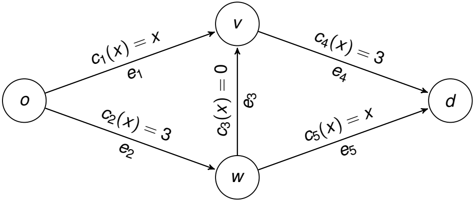

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$$C_i(s) = \sum_{e \in s_i} c_e(x_e).$$

Atomic selfish routing game (cont'd)

Suppose we have $n = 4$ players and edges $E = \{e_1, \ldots, e_4\}$.

- Remember that player places one unit of flow on a path.
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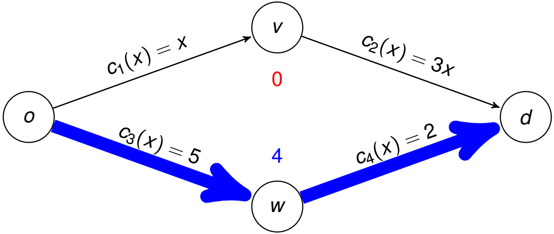

$s = (B, B, B, B)$

$$C_1(s) = 5 + 2 = 7$$
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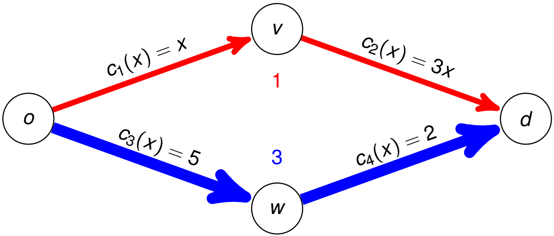

$s = (T, B, B, B)$

$$C_1(s) = 1 + 3 \cdot 1 = 4$$
$$C_2(s) = 5 + 2 = 7$$
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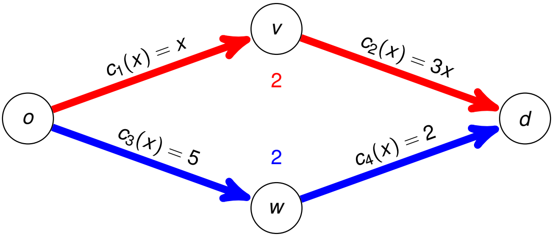

$s = (T, T, B, B)$

$$C_1(s) = 2 + 3 \cdot 2 = 8$$
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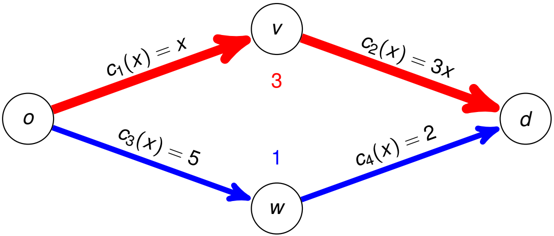

$s = (T, T, T, B)$

$$C_1(s) = 3 + 3 \cdot 3 = 12$$
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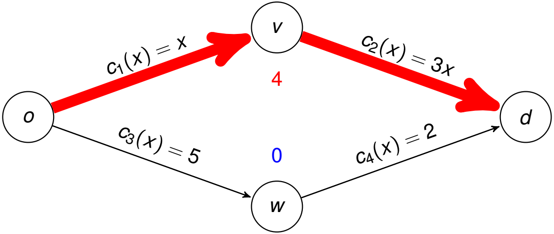

$s = (T, T, T, T)$

$$C_1(s) = 4 + 3 \cdot 4 = 16$$
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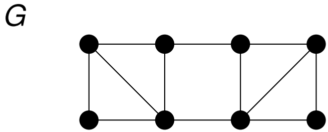
Broadcast game (example)

Given is undirected graph $G = (V, E)$.

- Edges $e \in E$ are resources with cost function c_e.
- Players place one unit of unsplittable load on spanning tree of G.
  - Spanning trees are the strategies of the players.



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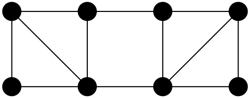

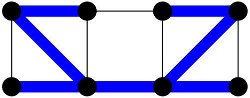

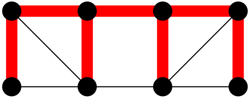

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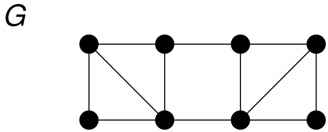

Example of base (graphic) matroid congestion game.

Pure Nash equilibrium

We will focus on pure Nash equilibria in congestion games.

Definition (Pure Nash equilibrium (PNE))

A strategy profile $s \in \times_i \mathcal{S}_i$ is a pure Nash equilibrium if for every $i \in N$,

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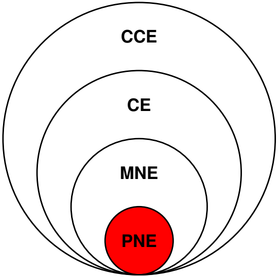

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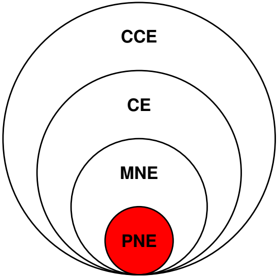

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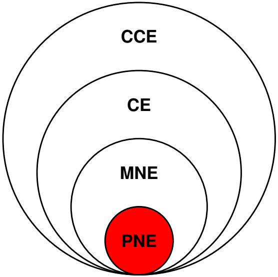

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ALGORITHM 4: Better response dynamics

Input : Strategy profile $s^0 \in \times_i \mathcal{S}_i$.
Output: Pure Nash equilibrium s^*.

$k = 0$.
while s^k is not a pure Nash equilibrium **do**
  Select player $i \in N$ and $s_i' \in \mathcal{S}_i$ such that $C_i(s_i', s_{-i}) < C_i(s)$.
  $s^{k+1} \leftarrow (s_i', s_{-i}^k)$.
  $k \leftarrow k + 1$.
end
return $s^* \leftarrow s^k$

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What is the potential function Φ?

Rosenthal's potential

The Rosenthal (potential) function $\Phi : \times_i \mathcal{S}_i \to \mathbb{R}$ is given by

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

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Rosenthal's potential satisfies, for every $i \in N$ and $s_i' \in \mathcal{S}_i$,

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- *Proof (sketch) on Slide 12 for symmetric singleton games*.
- Exercise: Generalize the proof to general congestion games.

Rosenthal's potential (example)

Remember, for strategy profile s,

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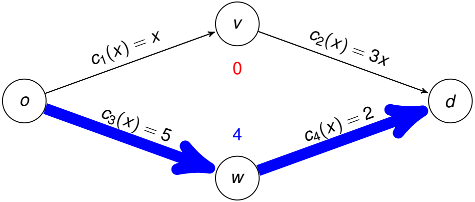

$$\begin{aligned}\Phi(s) &= 0 \\ &+ 0 \\ &+ [c_3(1) + c_3(2) + c_3(3) + c_3(4)] \\ &+ [c_4(1) + c_4(2) + c_4(3) + c_4(4)] \\ &= 28\end{aligned}$$

$$\begin{aligned}C_1(s) &= 5 + 2 = 7 \\ C_2(s) &= 5 + 2 = 7 \\ C_3(s) &= 5 + 2 = 7 \\ C_4(s) &= 5 + 2 = 7\end{aligned}$$

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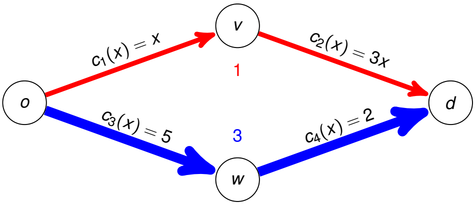

$$\begin{aligned}\Phi(s) &= [c_1(1)] \\ &+ [c_2(1)] \\ &+ [c_3(1) + c_3(2) + c_3(3)] \\ &+ [c_4(1) + c_4(2) + c_4(3)] \\ &= 25\end{aligned}$$

$$C_1(s) = 1 + 3 \cdot 1 = 4$$
$$C_2(s) = 5 + 2 = 7$$
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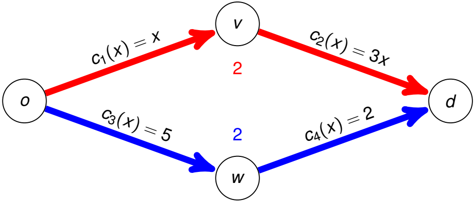

$$C_1(s) = 2 + 3 \cdot 2 = 8$$
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$$\begin{aligned}\Phi(s) &= [c_1(1) + c_1(2)] \\ &+ [c_2(1) + c_2(2)] \\ &+ [c_3(1) + c_3(2)] \\ &+ [c_4(1) + c_4(2)] \\ &= 26\end{aligned}$$

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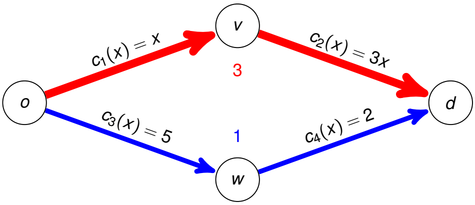

$$\begin{aligned}\Phi(s) &= [c_1(1) + c_1(2) + c_1(3)] \\ &\quad + [c_2(1) + c_2(2) + c_2(3)] \\ &\quad + [c_3(1)] \\ &\quad + [c_4(1)] \\ &= 31\end{aligned}$$

$$C_1(s) = 3 + 3 \cdot 3 = 12$$
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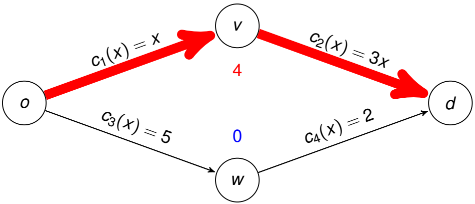

$$\begin{aligned}\Phi(s) &= [c_1(1) + c_1(2) + c_1(3) + c_1(4)] \\ &\quad + [c_2(1) + c_2(2) + c_2(3) + c_2(4)] \\ &\quad + 0 \\ &\quad + 0 \\ &= 40\end{aligned}$$

$$C_1(s) = 4 + 3 \cdot 4 = 16$$
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Symmetric singleton game $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ given by

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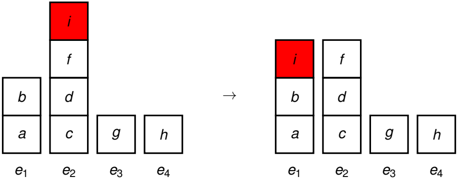

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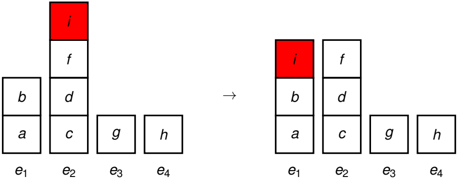

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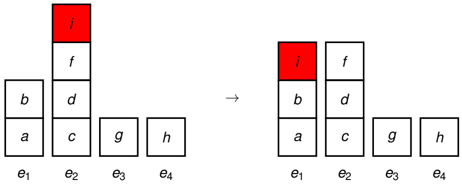

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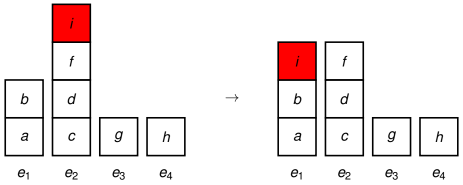

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Definition (Exact potential game)

Finite game $\Gamma = (N, (\mathcal{S}_i), (C_i))$ is exact potential game if there exists function $\Phi : \times_i \mathcal{S}_i \to \mathbb{R}$ such that

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Theorem

The class of congestion games is 'isomorphic' to the class of exact potential games.

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How to study computational complexity of PNE in congestion games?
*Interpret it as **local search problem** w.r.t. Rosenthal's potential.*

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ALGORITHM 7: Best response dynamics

Input : Strategy profile $s^0 \in \times_i \mathcal{S}_i$.
Output: Pure Nash equilibrium s^*.

$k = 0$.
while s^k is not a pure Nash equilibrium **do**
    Select player $i \in N$ and $s_i' \in \mathcal{S}_i$ such that

$$C_i(s_i', s_{-i}) = \min_{t_i \in \mathcal{S}_i} C_i(t_i, s_{-i}).$$

    $s^{k+1} \leftarrow (s_i', s_{-i}^k)$.
    $k \leftarrow k + 1$.
end
return $s^* \leftarrow s^k$

Some positive results to algorithmic questions

1. Special cases where response dynamics converge quickly:
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1. Special cases where response dynamics converge quickly:
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2. Special case where PNE can be computed by other means:
   - Symmetric network congestion games.

Singleton congestion games

Definition

A singleton congestion game $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ has the property that $\mathcal{S}_i \subseteq \{\{e_1,\}, \{e_2\}, \ldots, \{e_m\}\}$, i.e., every possible strategy consists of a single resource.

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For singleton congestion games, better response dynamics (BRD) terminate in at most $n^2 m$ steps (with n #players and m #resources).

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- $\Phi_{\max}, \Phi_{\min}$ are max. and min. attained by Φ, respectively.
  - For any strategy profile s, it holds that $\Phi_{\min} \leq \Phi(s) \leq \Phi_{\max}$.

Proof idea: Show that cost functions can be replaced by ‘nice’ (polynomially bounded, integer) cost functions while preserving improving moves.

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Step 1: Defining the 'nice' cost functions.
Consider $C = \bigcup_{e \in E} \{c_e(1), \ldots, c_e(n)\}$.

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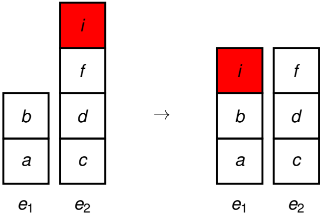

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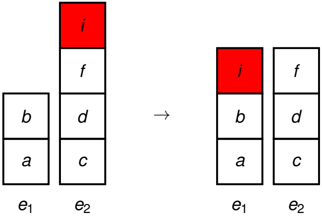

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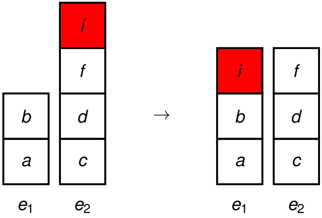

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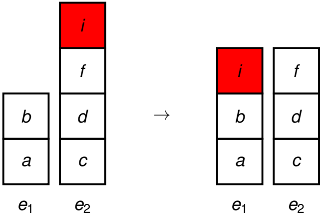

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Exercise: Show that this transformation fails for non-singleton congestion games (i.e., in general (1) is not true).

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Then apply lemma from Slide 17. □

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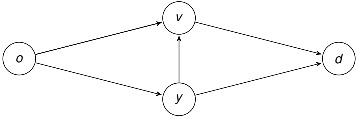

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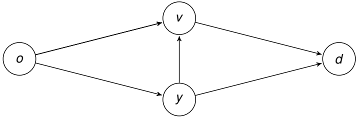

Resulting loads $x_e(s) = f_e$ satisfy the linear (in)equalities

$$\mathcal{F} = \Big\{ f \in \mathbb{R}^{|E|}_{\geq 0} : \quad \begin{array}{ll} \displaystyle\sum_{w:(w,v)\in E} f_{wv} = \sum_{w:(v,w)\in E} f_{vw} & \forall v \in V \setminus \{o, d\} \\ \displaystyle\sum_{w:(o,w)\in E} f_{ow} = n & \\ \displaystyle\sum_{w:(w,d)\in E} f_{wd} = n & \\ f_{vw} \geq 0 & \forall (v, w) \in E \Big\}. \end{array}$$

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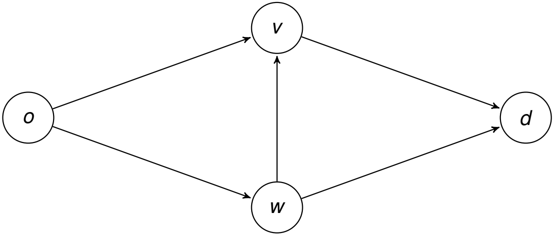
Remark

This high-level approach also works for other congestion games with some 'combinatorial' structure, e.g., (Del Pia-Michini-Ferris, 2015).

Minimum cost flow problem

Directed graph $G = (V, A)$ with origin o and destination d; flow size $n \in \mathbb{Q}$.

- Edge $e = (v, w) \in E$ has capacity u_{vw} and cost k_{vw}.



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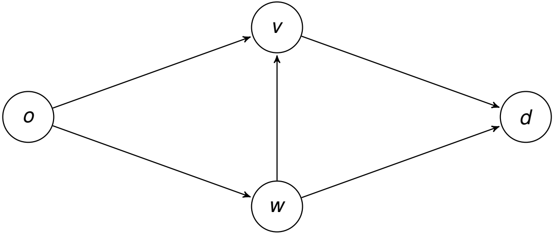

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Integral flow can be found in poly-time, when capacities are integral.

Reduction to min-cost flow problem (try yourself!)

Problem is that

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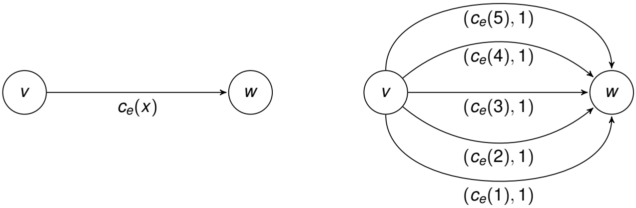

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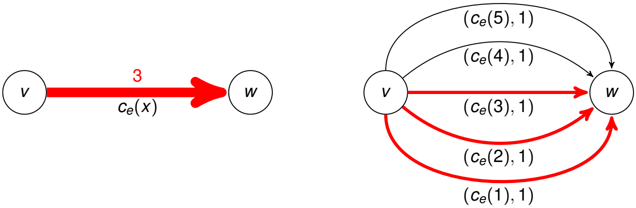

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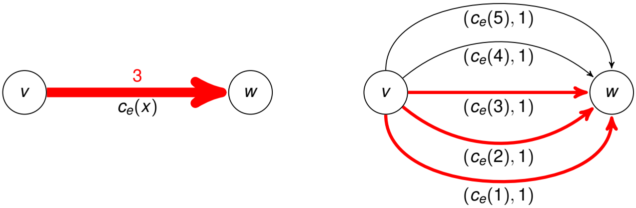

Every integral min-cost flow of size n in graph with copied edges corresponds to flow minimizing $\bar{\Phi}$.

Local search

High-level idea

Given function $f : S \to \mathbb{R}$, where S is a finite set.

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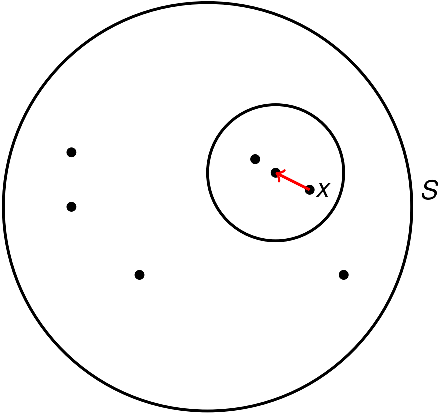

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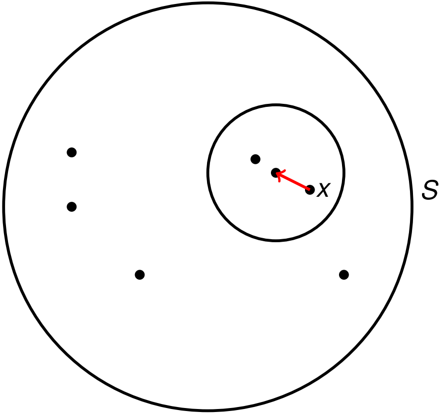

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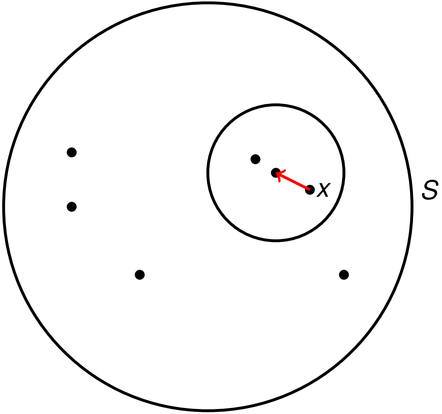

- Recall better response dynamics.
  - Essentially tries to find local improvement for Rosenthal's potential.

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- Set of instances $\mathcal{I}$;

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The procedure in which one repeatedly tries to find a better solution in the neighborhood is known as local search.

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Max-cut

Given undirected graph $G = (V, E)$ and weight function $w : E \to \mathbb{R}_{\geq 0}$, find partition $V = S \cup \bar{S}$ that maximizes

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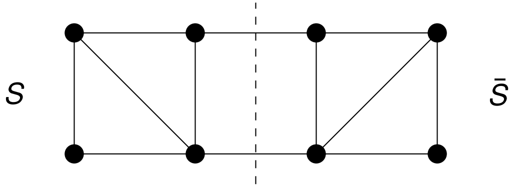

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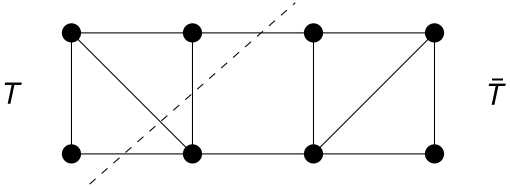

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- for all $I \in \mathcal{I}_1$: if S_2 is a local minimum of $f(I)$, then $g(S_2, I)$ is a local minimum of I.

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- f maps every instance $I \in \mathcal{I}_1$ of Π_1 to an instance $f(I) \in \mathcal{I}_2$ of Π_2;
- g maps every tuple (S_2, I) with $S_2 \in F_2(f(I))$ to a solution $S_1 \in F_1(I)$; (**Feasible solutions map to feasible solutions.**)
- for all $I \in \mathcal{I}_1$: if S_2 is a local minimum of $f(I)$, then $g(S_2, I)$ is a local minimum of I. (**Local minima map to local minima.**)

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Remark

The definition of PLS does not require you to solve a PLS(-complete) problem with local search.

From max-cut to PNE in congestion games

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- Reduction from Max-cut with FLIP neighborhoods.

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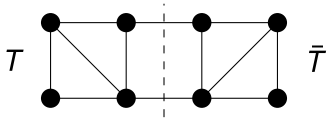

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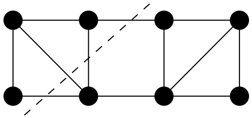
T  $\bar{T}$

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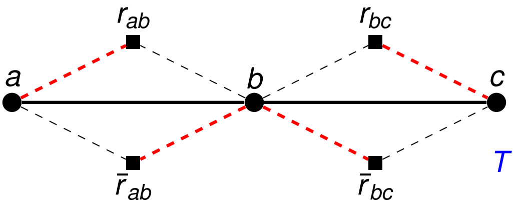

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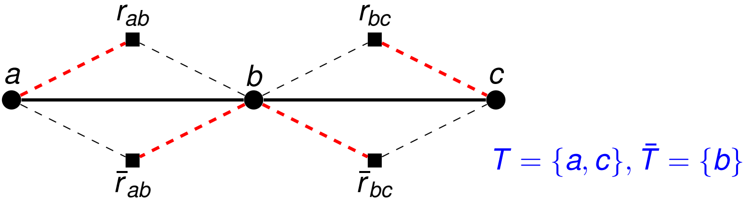

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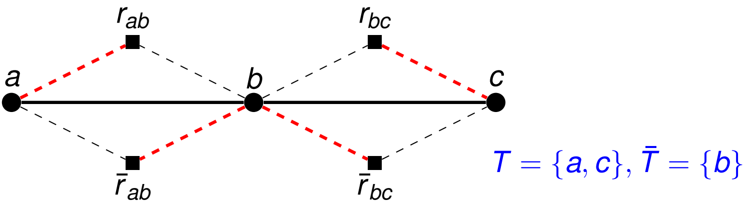

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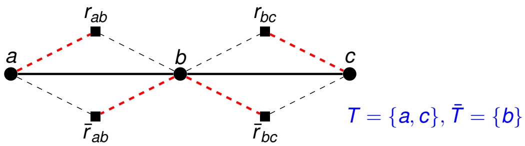

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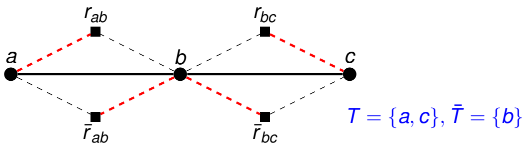

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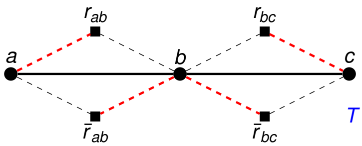

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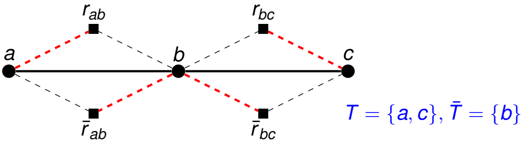

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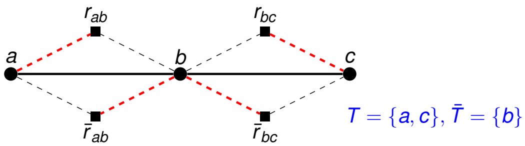

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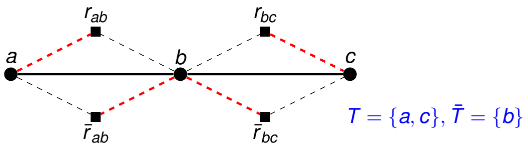

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PNEs of game are precisely locally min-uncuts/max-cuts!

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Big open question: Does (smoothed) local search for max-cut always converge in polynomial number of steps, for any graph *G*?