Topics in Algorithmic Game Theory and Economics

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Lecture 3
Congestion games II - Inefficiency of PNE

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Pure Nash equilibrium

We will focus on pure Nash equilibria in congestion games.

Definition (Pure Nash equilibrium (PNE))

A strategy profile $s \in \times_i \mathcal{S}_i$ is a pure Nash equilibrium if for every $i \in N$,

$$C_i(s_1, \ldots, s_i, \ldots, s_n) \leq C_i(s_1, \ldots, s_i', \ldots, s_n)$$

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From Lecture 2:

- *Computing PNE is PLS-complete problem in general.*
- *PNE can be computed efficiently in special case of symmetric network congestion games.*
- *Better response dynamics converge rapidly in singleton congestion games.*

Inefficiency of equilibria

Inefficiency of PNE

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How close is social cost of (pure) Nash equilibrium to that of a social optimum? Multiple answers, as equilibrium is in general not unique.

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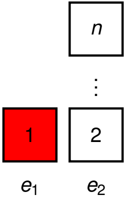
$$c_{e_1}(x) = 1 + \epsilon$$



$$c_{e_2}(x) = \begin{cases} 0 & x < n \\ 1 & x = n. \end{cases}$$

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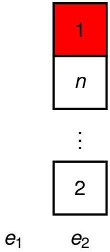

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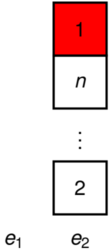

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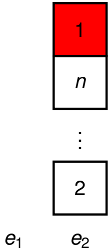

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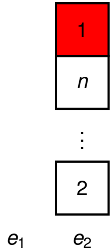

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Yes, if we make assumptions on the cost functions.

Polynomial cost functions

Polynomial cost functions are of the form

$$c_e(y) = \sum_{j=0}^{d} a_{j,e} y^d$$

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- Etc...

PoA of affine congestion game Γ

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PoA for affine congestion games

Theorem (Christodoulou and Koutsoupias (2005))

Let $\mathcal{G}$ be the set of all congestion games with cost functions of the form $c_e(y) = a_e y + b_e$ where $a_e, b_e \geq 0$. It holds that

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Remark

For polynomials of degree at most d, a tight bound is known as well. It grows roughly like $d^{d(1-o(1))}$.

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If Γ *is* (λ, μ)*-smooth and* $\mu < 1$*, then* $PoA(\Gamma) \leq \frac{\lambda}{1-\mu}$.

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$$C(s) = \sum_{i \in N} C_i(s_i, s_{-i}) \leq \sum_{i \in N} C_i(s_i^*, s_{-i}) \leq \lambda C(s^*) + \mu C(s).$$

Rearranging terms and exploiting that $\mu < 1$ proves the claim. □

Smoothness technique (cont'd)

Theorem (Roughgarden, 2009)

If Γ is (λ, μ)-smooth and $\mu < 1$, then PoA(Γ) $\leq \frac{\lambda}{1-\mu}$.

- Remember that we defined the PoA for pure Nash equilibria.

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Definition

The robust price of anarchy of a strategic game Γ is defined as

$$RPoA(\Gamma) = \inf \left\{ \frac{\lambda}{1-\mu} \ : \ \Gamma \text{ is } (\lambda, \mu)\text{-smooth with } \mu < 1 \right\}.$$

For a class $\mathcal{G}$ of games, we define

$$RPoA(\mathcal{G}) = \sup \{ RPoA(\Gamma) \ : \ \Gamma \in \mathcal{G} \}.$$

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- RPoA automatically extends to other equilibria types in hierarchy.
  - Mixed, correlated and coarse correlated equilibria.

PoA for affine congestion games (cont'd)

Theorem

The price of anarchy of affine congestion games is $\frac{5}{2}$.

PoA for affine congestion games (cont'd)

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For $t \in \times_i \mathcal{S}_i$, we have

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Fact

Let α and β be two non-negative integers. Then

$$\alpha(\beta + 1) \leq \frac{5}{3}\alpha^2 + \frac{1}{3}\beta^2.$$

Tightness holds for $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (1, 2)$.

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By applying this to every resource $e \in E$, we get

$$C(s) \leq \sum_{e \in E} x_e^*(x_e + 1) \leq \frac{5}{3} \sum_{e \in E} (x_e^*)^2 + \frac{1}{3} \sum_{e \in E} (x_e)^2 = \frac{5}{3} C(s^*) + \frac{1}{3} C(s). \quad \square$$

PoA lower bound for affine congestion games

Congestion game instance

- $N = \{1, 2, 3\}$.

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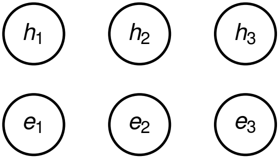

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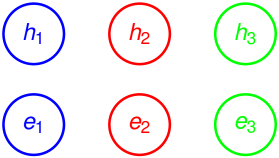

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Every player i plays $\{e_i, h_i\}$, which gives $C(s^*) = 6$.

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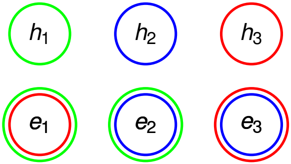

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Pure Nash equilibrium s

Every player i plays $\{e_{i-1}, e_{i+1}, h_{i+1}\}$, which gives $C(s) = 15$.

Extension to other equilibrium types in hierarchy

Definition

Γ is called (λ, μ)-smooth if for any two strategy profiles $s, s^* \in \times_i \mathcal{S}_i$,

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Remember that $PNE \subseteq MNE \subseteq CE \subseteq CCE$.

- Pure, mixed, correlated and coarse correlated equilibria.

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Definition (PoA, general)

For $\mathrm{EQ} \in \{\mathrm{PNE}, \mathrm{MNE}, \mathrm{CE}, \mathrm{CCE}\}$, we define

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Theorem (Extension)

If the game Γ is (λ, μ)-smooth, then $RPoA_{EQ}(\Gamma) \leq \lambda/(1-\mu)$.

Suffices to show extension theorem for coarse correlated equilibria.

Definition (Coarse correlated equilibrium (CCE))

A distribution σ on $\times_i \mathcal{S}_i$ is a coarse correlated equilibrium if for every $i \in N$, and every unilateral deviation $s'_i \in \mathcal{S}_i$, it holds that

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Suffices to show extension theorem for coarse correlated equilibria.

Definition (Coarse correlated equilibrium (CCE))

A distribution σ on $\times_i \mathcal{S}_i$ is a coarse correlated equilibrium if for every $i \in N$, and every unilateral deviation $s_i' \in \mathcal{S}_i$, it holds that

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Price of Anarchy of affine congestion games is at most $5/2$ *and this bound is tight.*

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- Same is true for Price of Stability bounds (up next).

Price of Stability for affine congestion game Γ

$$\text{PoS}(\Gamma) = \frac{\min_{s \in \text{PNE}} C(s)}{\min_{t \in \times_i \mathcal{S}_i} C(t)}$$

Some intuition

For bounding the PoS, we need to do something else than just use the inequalities defining a PNE.

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For congestion games $(N, E, (\mathcal{S}_i), (c_e))$, recall that Rosenthal's potential $\Phi : \times_i \mathcal{S}_i \to \mathbb{R}$ is given by

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- Strategy profile s is global minimizer if $\Phi(s) \leq \Phi(t)$ for all $t \in \times_i \mathcal{S}_i$.

Potential function approach for bounding PoS

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Let Γ *be congestion game and* $C : \times_i \mathcal{S}_i \to \mathbb{R}_{>0}$ *social cost function given by* $C(s) = \sum_i C_i(s)$.

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PoS for affine congestion games

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Remark

Can be improved to tight bound of $1 + 1/\sqrt{3} \approx 1.577$.

Tight PoS bound

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PNE inequalities

$$\sum_{e \in E} x_e^2 = \sum_i C_i(s) \leq \sum_i C_i(s_i^*, s_{-i}) \leq \sum_{e \in E} x_e^*(x_e + 1)$$

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Overview for affine congestion games

Theorem

Price of Anarchy for affine congestion games is $5/2$.

- Relatively simple lower bound construction showing tightness.
- Proof extends to other equilibrium types in hierarchy by means of (λ, μ)-smoothness technique.

Theorem

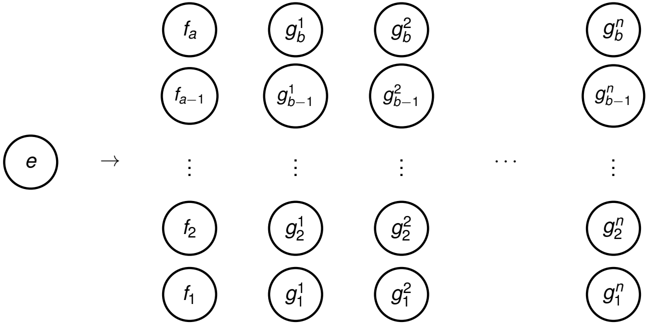
Price of Stability for affine congestion games is $1 + 1/\sqrt{3} \approx 1.577$.

- Tightness examples are more involved.
- Proof does not extend to other equilibrium types in hierarchy.

About the assumption $c_e(x) = x$

Suppose that $c_e(x) = a_e x + b_e$ with $a_e, b_e \in \mathbb{N}$. (Same works for $\mathbb{Q}$.)

- We write $a = a_e,\ b = b_e$.



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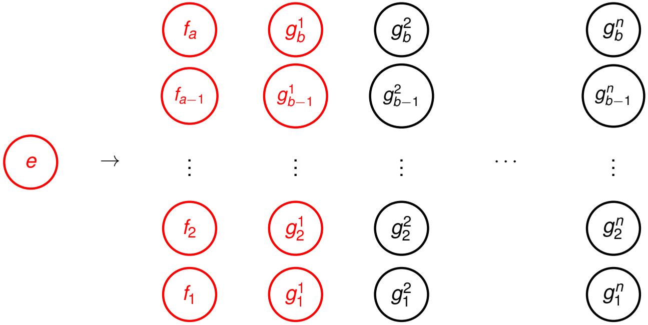


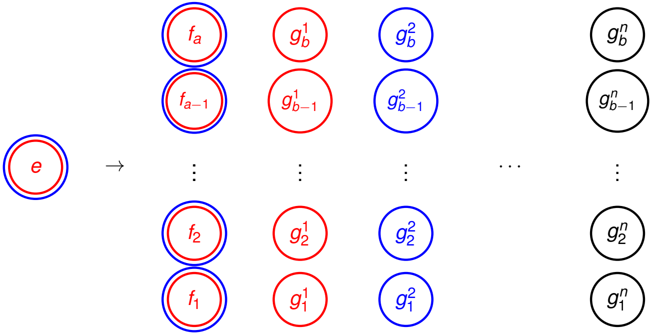
If player i uses e in some of its strategies, replace it by set

$$K_i(e) = \{f_1, \dots, f_a, g_1^i, \dots, g_b^i\}.$$

About the assumption $c_e(x) = x$

Suppose that $c_e(x) = a_e x + b_e$ with $a_e, b_e \in \mathbb{N}$. (Same works for $\mathbb{Q}$.)

- We write $a = a_e,\ b = b_e$.



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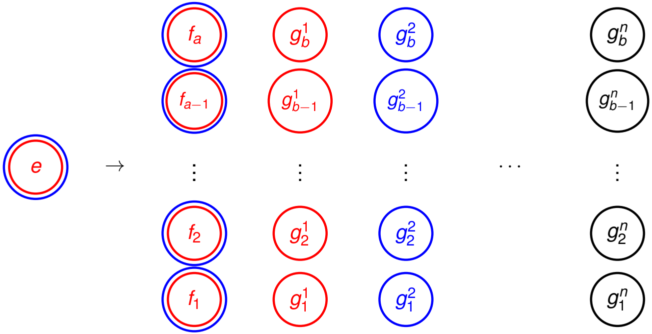

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Cost incurred on resources $K_i(e)$ ***is precisely*** $ax_e + b$ ***if load is*** x_e.