Topics in Algorithmic Game Theory and Economics

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November 25, 2020

Lecture 3
Congestion games II - Inefficiency of PNE

Congestion games

Congestion game Γ:

- Set of players $N = \{1, \dots, n\}$.
- Set of resources $E = \{e_1, \dots, e_m\}$.
- Strategy set $\mathcal{S}_i \subseteq 2^E$ for all $i \in N$.
  - *(s, t)-paths in directed graph.*
- Cost function $c_e : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for $e \in E$.

Player places one unit of unsplittable load on a strategy with goal of minimizing her cost.

For strategy profile $s = (s_1, \dots, s_n) \in S_1 \times S_2 \times \cdots \times S_n = \times_i \mathcal{S}_i$,

$$C_i(s) = \sum_{e \in s_i} c_e(x_e),$$

where $x_e = x_e(s)$ is the number of players using $e \in E$.

Pure Nash equilibrium

We will focus on pure Nash equilibria in congestion games.

Definition (Pure Nash equilibrium (PNE))

A strategy profile $s \in \times_i \mathcal{S}_i$ is a pure Nash equilibrium if for every $i \in N$,

$$C_i(s_1, \ldots, s_i, \ldots, s_n) \leq C_i(s_1, \ldots, s_i', \ldots, s_n)$$

for every $s_i' \in \mathcal{S}_i$. *In short,* $C_i(s) \leq C_i(s_i', s_{-i})$.

From Lecture 2:

- *Computing PNE is PLS-complete problem in general.*
- *PNE can be computed efficiently in special case of symmetric network congestion games.*
- *Better response dynamics converge rapidly in singleton congestion games.*

Inefficiency of equilibria

Inefficiency of PNE

Let $\Gamma = (N, (\mathcal{S}_i), (C_i))$ be a finite game and

$$C : \times_i \mathcal{S}_i \to \mathbb{R}_{>0}$$

a social cost objective assigning cost to every strategy profile.

- Unless specified otherwise, we consider total player cost

$$C(s) = \sum_{i \in N} C_i(s).$$

Definition (Social optimum)

A social optimum $s^* \in \times_i \mathcal{S}_i$ is a strategy profile that minimizes C, i.e.,

$$C(s^*) = \min\{C(s) : s \in \times_i \mathcal{S}_i\}.$$

How close is social cost of (pure) Nash equilibrium to that of a social optimum? Multiple answers, as equilibrium is in general not unique.

Inefficiency measures

Definition (Price of Anarchy/Stability)

Price of Anarchy (PoA) and Price of Stability (PoS) of finite game Γ are given by

$$\mathrm{PoA}(\Gamma) = \frac{\max_{s \in \mathrm{PNE}} C(s)}{\min_{t \in \times_i \mathcal{S}_i} C(t)} \quad \text{and} \quad \mathrm{PoS}(\Gamma) = \frac{\min_{s \in \mathrm{PNE}} C(s)}{\min_{t \in \times_i \mathcal{S}_i} C(t)},$$

where $\mathrm{PNE} = \mathrm{PNE}(\Gamma)$ is the set of **pure** Nash equilibria of Γ. For a (possibly infinite) class of games $\mathcal{G}$, we have

$$\mathrm{PoA}(\mathcal{G}) = \sup_{\Gamma \in \mathcal{G}} \mathrm{PoA}(\Gamma) \quad \text{and} \quad \mathrm{PoS}(\mathcal{G}) = \sup_{\Gamma \in \mathcal{G}} \mathrm{PoS}(\Gamma).$$

- PoA measures worst-case inefficiency due to strategic behaviour.
- PoS measures minimal inefficiency due to strategic behaviour.

We always have

$$\mathrm{PoS}(\Gamma) \leq \mathrm{PoA}(\Gamma).$$

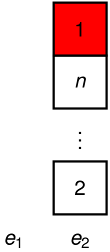
Inefficiency in congestion games

Both PoA and PoS can be unbounded already in simple games.

- Unbounded in number of players $n = |N|$.

Consider the following game Γ.

$$c_{e_1}(x) = 1 + \epsilon \qquad\qquad c_{e_2}(x) = \begin{cases} 0 & x < n \\ 1 & x = n. \end{cases}$$



$$C(s) = \sum_i C_i(s) = 1 + (1 + \cdots + 1) = n$$

$$\text{PoA}(\Gamma) = \text{PoS}(\Gamma) = \tfrac{n}{1+\epsilon}$$

Are there classes of games where PoA/PoS is constant (for any n)?

Yes, if we make assumptions on the cost functions.

Polynomial cost functions

Polynomial cost functions are of the form

$$c_e(y) = \sum_{j=0}^{d} a_{j,e} y^d$$

with $a_{j,e} \geq 0$ for all $j = 0, \ldots, d$ and $e \in E$.

- Affine when $d = 1$, i.e.,

$$c_e(y) = a_{e,1} y + a_{e,0}.$$

- Quadratic when $d = 2$, i.e.,

$$c_e(y) = a_{e,2} y^2 + a_{e,1} y + a_{e,0}.$$

- Etc...

PoA of affine congestion game Γ

$$\mathrm{PoA}(\Gamma) = \frac{\max_{s \in \mathrm{PNE}} C(s)}{\min_{t \in \times_i \mathcal{S}_i} C(t)}$$

PoA for affine congestion games

Theorem (Christodoulou and Koutsoupias (2005))

Let $\mathcal{G}$ be the set of all congestion games with cost functions of the form $c_e(y) = a_e y + b_e$ where $a_e, b_e \geq 0$. It holds that

$$PoA(\mathcal{G}) = \frac{5}{2}.$$

- (Asymptotic) tightness holds even for the special class of symmetric network congestion games.
  - De Keijzer et al. (2015)
- Does not hold for, e.g., symmetric singleton congestion games.
  - Tight bound of $4/3$ is known for this class.

Remark

For polynomials of degree at most d, a tight bound is known as well. It grows roughly like $d^{d(1-o(1))}$.

Smoothness technique

Let $\Gamma = (N, (S_i)_{i\in N}, (C_i)_{i\in N})$ be a finite (cost minimization) game with social cost $C(s) = \sum_{i\in N} C_i(s)$.

Definition

Γ is called (λ, μ)-smooth if for any two strategy profiles $s, s^* \in \times_i \mathcal{S}_i$,

$$\sum_{i\in N} C_i(s_i^*, s_{-i}) \leq \lambda C(s^*) + \mu C(s). \tag{1}$$

Theorem (Roughgarden, 2009)

If Γ is (λ, μ)-smooth and $\mu < 1$, then $PoA(\Gamma) \leq \frac{\lambda}{1-\mu}$.

Proof: Let s be a pure Nash equilibrium and s^* a social optimum. Then

$$C(s) = \sum_{i\in N} C_i(s_i, s_{-i}) \leq \sum_{i\in N} C_i(s_i^*, s_{-i}) \leq \lambda C(s^*) + \mu C(s).$$

Rearranging terms and exploiting that $\mu < 1$ proves the claim. □

Smoothness technique (cont'd)

Theorem (Roughgarden, 2009)

If Γ is (λ, μ)-smooth and $\mu < 1$, then PoA(Γ) $\leq \frac{\lambda}{1-\mu}$.

- Remember that we defined the PoA for pure Nash equilibria.

Definition

The robust price of anarchy of a strategic game Γ is defined as

$$RPoA(\Gamma) = \inf \left\{ \frac{\lambda}{1-\mu} \ : \ \Gamma \text{ is } (\lambda, \mu)\text{-smooth with } \mu < 1 \right\}.$$

For a class $\mathcal{G}$ of games, we define

$$RPoA(\mathcal{G}) = \sup \left\{ RPoA(\Gamma) \ : \ \Gamma \in \mathcal{G} \right\}.$$

- RPoA automatically extends to other equilibria types in hierarchy.
  - Mixed, correlated and coarse correlated equilibria.

PoA for affine congestion games (cont'd)

Theorem

The price of anarchy of affine congestion games is $\frac{5}{2}$.

Proof: Let s be PNE and s^* a social optimum. We will show that

$$\sum_{i \in N} C_i(s_i^*, s_{-i}) \leq \frac{5}{3} C(s^*) + \frac{1}{3} C(s),$$

i.e., that the game is $(5/3, 1/3)$-smooth.

- Assume (w.l.o.g.) that $c_e(x) = x$ for every $e \in E$.
- Write $x_e = x_e(s)$ and $x_e^* = x_e(s^*)$.

$$\begin{aligned}
C(s) &= \textstyle\sum_{i \in N} C_i(s) \\
&\leq \textstyle\sum_{i \in N} C_i(s_i^*, s_{-i}) && \textit{(using PNE definition)} \\
&= \textstyle\sum_{i \in N} \sum_{e \in s_i^*} c_e(x_e(s_i^*, s_{-i})) \\
&= \textstyle\sum_{i \in N} \sum_{e \in s_i^*} x_e(s_i^*, s_{-i}) && (c_e(x) = x) \\
&\leq \textstyle\sum_{i \in N} \sum_{e \in s_i^*} x_e + 1 \\
&= \textstyle\sum_{e \in E} \sum_{i : e \in s_i^*} x_e + 1 && \textit{(interchanging summation)} \\
&= \textstyle\sum_{e \in E} x_e^* (x_e + 1)
\end{aligned}$$

For $t \in \times_i \mathcal{S}_i$, we have

$$C(t) = \sum_i C_i(t) = \sum_{e \in E} y_e c_e(y_e) \text{ with } y_e = y_e(t)$$

With the assumption $c_e(x) = x$ for all $e \in E$, it suffices to show that

$$\sum_{e \in E} x_e^*(x_e + 1) \leq \frac{5}{3} \sum_{e \in E} (x_e^*)^2 + \frac{1}{3} \sum_{e \in E} (x_e)^2.$$

Fact

Let α and β be two non-negative integers. Then

$$\alpha(\beta + 1) \leq \frac{5}{3}\alpha^2 + \frac{1}{3}\beta^2.$$

Tightness holds for $(\alpha, \beta) = (1, 1)$ and $(\alpha, \beta) = (1, 2)$.

By applying this to every resource $e \in E$, we get

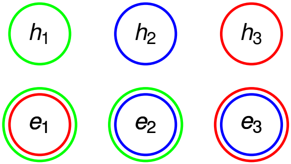
$$C(s) \leq \sum_{e \in E} x_e^*(x_e + 1) \leq \frac{5}{3} \sum_{e \in E} (x_e^*)^2 + \frac{1}{3} \sum_{e \in E} (x_e)^2 = \frac{5}{3} C(s^*) + \frac{1}{3} C(s). \quad \square$$

PoA lower bound for affine congestion games

Congestion game instance

- $N = \{1, 2, 3\}$.
- $E = E_1 \cup E_2$, where $E_1 = \{e_1, e_2, e_3\}$ and $E_2 = \{h_1, h_2, h_3\}$.
- Cost function $c_e(x) = x$ for every $e \in E$.
- Each player i has two strategies (modulo 3):

$$\mathcal{S}_i = \{\{e_i, h_i\}, \{e_{i-1}, e_{i+1}, h_{i+1}\}\}.$$



Social optimum s^*

Every player i plays $\{e_i, h_i\}$, which gives $C(s^*) = 6$.

Pure Nash equilibrium s

Every player i plays $\{e_{i-1}, e_{i+1}, h_{i+1}\}$, which gives $C(s) = 15$.

Extension to other equilibrium types in hierarchy

Definition

Γ is called (λ, μ)-smooth if for any two strategy profiles $s, s^* \in \times_i \mathcal{S}_i$,

$$\sum_{i \in N} C_i(s_i^*, s_{-i}) \leq \lambda C(s^*) + \mu C(s). \quad (2)$$

Remember that $PNE \subseteq MNE \subseteq CE \subseteq CCE$.

- Pure, mixed, correlated and coarse correlated equilibria.

Definition (PoA, general)

For EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$, we define

$$\text{PoA}_{EQ}(\Gamma) = \frac{\max_{\sigma \in \text{EQ}} \mathbb{E}_{s \sim \sigma}\left[C(s)\right]}{\min_{t \in \times_i \mathcal{S}_i} C(t)}$$

where for $s \in \times_i \mathcal{S}_i$, one has $\mathbb{E}_{s \sim \sigma}\left[C_i(s)\right] = \sum_{s \in \times_i \mathcal{S}_i} \sigma(s) C_i(s)$.

Theorem (Extension)

If the game Γ is (λ, μ)-smooth, then $RPoA_{EQ}(\Gamma) \leq \lambda/(1-\mu)$.

Suffices to show extension theorem for coarse correlated equilibria.

Definition (Coarse correlated equilibrium (CCE))

A distribution σ on $\times_i \mathcal{S}_i$ is a coarse correlated equilibrium if for every $i \in N$, and every unilateral deviation $s_i' \in \mathcal{S}_i$, it holds that

$$\mathbb{E}_{s\sim\sigma}\left[C_i(s)\right] \leq \mathbb{E}_{s\sim\sigma}\left[C_i(s_i', s_{-i})\right].$$

Proof (extension theorem): Let s be CCE and s^* social optimum. Then

$$\begin{aligned}
\mathbb{E}_{s\sim\sigma}\left[C(s)\right] &= \mathbb{E}_{s\sim\sigma}\left[\sum_i C_i(s)\right] \\
&= \sum_i \mathbb{E}_{s\sim\sigma}\left[C_i(s)\right] && \text{(lin. of expectation)} \\
&\leq \sum_i \mathbb{E}_{s\sim\sigma}\left[C_i(s_i^*, s_{-i})\right] && \text{(CCE definition)} \\
&= \mathbb{E}_{s\sim\sigma}\left[\sum_i C_i(s_i^*, s_{-i})\right] \\
&\leq \mathbb{E}_{s\sim\sigma}\left[\lambda C(s^*) + \mu C(s)\right] && \text{(smoothness)} \\
&= \lambda C(s^*) + \mu \cdot \mathbb{E}_{s\sim\sigma}\left[C(s)\right] && \square
\end{aligned}$$

Summary

Theorem

Price of Anarchy of affine congestion games is at most $5/2$ and this bound is tight.

- Also holds for other equilibrium types in hierarchy.
- Extendability is proved using the smoothness framework.

Note that smoothness framework (and extension theorem) apply to finite games in general, and not only congestion games.

Remark

The smoothness framework cannot always be applied. For example, a tight bound on the Price of Anarchy of $4/3$ is known for class of symmetric singleton congestion games, but this bound does not extend to more general equilibrium types.

- Same is true for Price of Stability bounds (up next).

Price of Stability for affine congestion game Γ

$$\mathrm{PoS}(\Gamma) = \frac{\min_{s \in \mathrm{PNE}} C(s)}{\min_{t \in \times_i \mathcal{S}_i} C(t)}$$

Some intuition

For bounding the PoS, we need to do something else than just use the inequalities defining a PNE.

For congestion games $(N, E, (\mathcal{S}_i), (c_e))$, recall that Rosenthal's potential $\Phi : \times_i \mathcal{S}_i \to \mathbb{R}$ is given by

$$\Phi(s) = \sum_{e \in E} \sum_{k=1}^{x_e(s)} c_e(k).$$

We have seen that PNEs are local minima of Rosenthal's potential (w.r.t the unilateral deviation neighbourhood). Do some local minima have better social cost than others?

Informal "rule of thumb" (not true in general):
PNE s that is a global minimizer of Rosenthal's potential has better social cost $C(s) = \sum_i C_i(s) = \sum_{e \in E} x_e c_e(x_e)$.

- Strategy profile s is global minimizer if $\Phi(s) \leq \Phi(t)$ for all $t \in \times_i \mathcal{S}_i$.

Potential function approach for bounding PoS

Theorem

Let Γ be congestion game and $C : \times_i \mathcal{S}_i \to \mathbb{R}_{>0}$ social cost function given by $C(s) = \sum_i C_i(s)$. Let $\alpha, \beta > 0$ be such that Rosenthal's potential Φ satisfies

$$C(t)/\alpha \leq \Phi(t) \leq \beta C(t)$$

for every $t \in \times_i \mathcal{S}_i$. Then $PoS(\Gamma) \leq \alpha\beta$.

Proof: Let s be a global minimizer of Φ. Note that s is then a pure Nash equilibrium. Let s^* be social optimum. We have

$$C(s) \leq \alpha\Phi(s) \leq \alpha\Phi(s^*) \leq \alpha\beta C(s^*).$$

□

PoS for affine congestion games

Theorem

The price of stability of affine congestion games is at most 2.

Proof: Let $t \in \times_i \mathcal{S}_i$ be a strategy profile. Assume again (w.l.o.g.) that $c_e(x) = x$ and write $x_e = x_e(t)$. Then

$$\begin{aligned}\Phi(t) = \sum_{e \in E} \sum_{k=1}^{x_e} c_e(k) = \sum_{e \in E} \sum_{k=1}^{x_e} k \quad &= \sum_{e \in E} \frac{1}{2} x_e(x_e + 1) \\ &= \frac{1}{2} C(t) + \frac{1}{2} \sum_{e \in E} x_e \\ &\leq \frac{1}{2} C(t) + \frac{1}{2} C(t).\end{aligned}$$

This implies that $\frac{1}{2} C(t) \leq \Phi(t) \leq C(t)$. □

Remark

Can be improved to tight bound of $1 + 1/\sqrt{3} \approx 1.577$.

Tight PoS bound

How to get tight PoS bound? Combine inequalities defining PNE and global minimizer inequality.

Let s be global minimizer Φ and let s^* be social optimum.

PNE inequalities

$$\gamma \times \left[\sum_{e \in E} x_e^2 = \sum_i C_i(s) \leq \sum_i C_i(s_i^*, s_{-i}) \leq \sum_{e \in E} x_e^*(x_e + 1) \right]$$

Global minimizer inequality

$$(1-\gamma) \times \left[\sum_{e \in E} x_e^2 = 2\Phi(s) \leq 2\Phi(s^*) = \sum_{e \in E} x_e^*(x_e^* + 1) - x_e \right]$$

Find constants $\lambda(\gamma),\ \mu(\gamma)$ such that

$$\begin{aligned} C(s) &\leq \sum_{e \in E} \gamma x_e^*(x_e + 1) + \sum_{e \in E} (1-\gamma)[x_e^*(x_e^* + 1) - x_e] \\ &\leq \lambda(\gamma) C(s^*) + \mu(\gamma) C(s) \quad \text{and optimize over } \gamma \text{...} \end{aligned}$$

Overview for affine congestion games

Theorem

Price of Anarchy for affine congestion games is $5/2$.

- Relatively simple lower bound construction showing tightness.
- Proof extends to other equilibrium types in hierarchy by means of (λ, μ)-smoothness technique.

Theorem

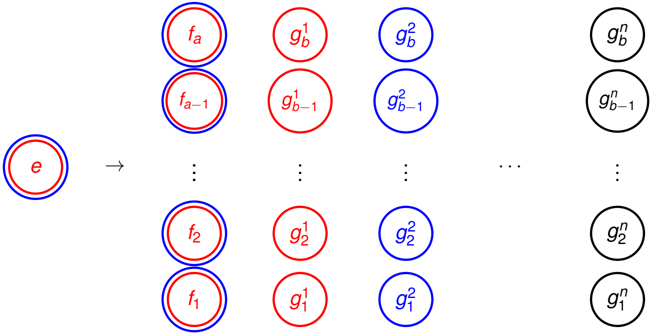
Price of Stability for affine congestion games is $1 + 1/\sqrt{3} \approx 1.577$.

- Tightness examples are more involved.
- Proof does not extend to other equilibrium types in hierarchy.

About the assumption $c_e(x) = x$

Suppose that $c_e(x) = a_e x + b_e$ with $a_e, b_e \in \mathbb{N}$. (Same works for $\mathbb{Q}$.)

- We write $a = a_e,\ b = b_e$.



If player i uses e in some of its strategies, replace it by set

$$K_i(e) = \{f_1, \dots, f_a, g_1^i, \dots, g_b^i\}.$$

***Cost incurred on resources** $K_i(e)$ **is precisely** $ax_e + b$ **if load is** x_e.*