Topics in Algorithmic Game Theory and Economics

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December 2, 2020

Lecture 4
Finite games - Existence and Computation of MNE

Finite game

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  - Mixed strategies $\sigma_A = (1/2, 1/2)$ and $\sigma_B = (1/2, 1/2)$.

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Example

Strategies of Alice and Bob are given by:

$$\begin{aligned} \Delta_{\text{Alice}} &= \{(x_1, x_2) : x_1 + x_2 = 1, x_1, x_2 \geq 0\}, \\ \Delta_{\text{Bob}} &= \{(y_1, y_2, y_3) : y_1 + y_2 + y_3 = 1, y_1, y_2, y_3 \geq 0\}. \end{aligned}$$

	b_1	b_2	b_3
a_1	$(0, 2)$	$(1, 0)$	$(2, 1)$
a_2	$(3, 0)$	$(0, 1)$	$(1, 4)$

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\Delta_{\text{A(lice)}} &= \{(x_1,\ldots,x_m) : \textstyle\sum_i x_i = 1, x_i \geq 0 \text{ for } i = 1,\ldots,m\},\\
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For $x \in \Delta_A, y \in \Delta_B$, we get product distribution $\sigma_{x,y} : \mathcal{S}_A \times \mathcal{S}_B \to [0,1]$ over strategy profiles,

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Example (cont'd)

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$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

	b_1	b_2	b_3
a_1	$(0, 2)$	$(1, 0)$	$(2, 1)$
a_2	$(3, 0)$	$(0, 1)$	$(1, 4)$

Expected cost under mixed strategies $x \in \Delta_A, y \in \Delta_B$ is then

$$C_{\text{Alice}}(x, y) = x^T A y = \sum_{k=1}^{m} \sum_{\ell=1}^{n} A_{k\ell} x_k y_\ell, \quad C_{\text{Bob}}(x, y) = x^T B y = \sum_{k=1}^{m} \sum_{\ell=1}^{n} B_{k\ell} x_k y_\ell$$

Short overview

Two-player game (A, B) is given by matrices $A, B \in \mathbb{R}^{m \times n}$, with player Alice choosing mixed strategy x over rows, and player Bob mixed strategy y over columns.

Matrix representation

Matrix representation of cost functions $C_i : \Delta_A \times \Delta_B \to \mathbb{R}$ for $i \in \{\text{Alice}, \text{Bob}\}$ given by $A, B \in \mathbb{R}^{m \times n}$ defined as

$$A_{k\ell} = C_A(a_k, b_\ell) \text{ and } B_{k\ell} = C_B(a_k, b_\ell) \text{ for } k = 1, \ldots, m \text{ and } \ell = 1, \ldots, n.$$

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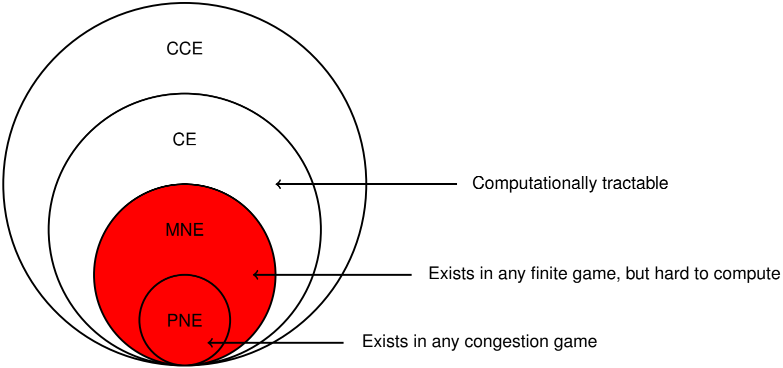
Mixed Nash equilibrium

Hierarchy of equilibrium concepts



Mixed Nash equilibrium (2-player case)

For two-player game (A, B), we have

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Definition (Mixed Nash equilibrium)

Pair $(x^*, y^*) \in \Delta_A \times \Delta_B$ is mixed Nash equilibrium (MNE) if neither Alice nor Bob can deviate to other mixed strategy and improve cost:

$$\begin{aligned} C_A(x^*, y^*) &\le C_A(x', y^*) \quad \forall x' \in \Delta_A \\ C_B(x^*, y^*) &\le C_B(x^*, y') \quad \forall y' \in \Delta_B \end{aligned}$$

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For $\epsilon > 0$, pair (x^*, y^*) is ϵ-approximate MNE (or simply ϵ-MNE) if

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Example

Alice has $\mathcal{S}_A = \{a_1, a_2\}$ and $\mathcal{S}_B = \{b_1, b_2, b_3\}$.

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 4 & 2 \\ 2 & 0 & 4 \end{pmatrix}.$$

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In MNE, players only have positive probability on rows/columns that minimize expected cost per row/column (given other's strategy).

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Finally, we write $e^k \in \Delta_A$ for pure strategy in which Alice plays $a_k \in \mathcal{S}_A$ with probability 1.

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- Exercise: Prove that this definition is equivalent to that on Slide 8.

Mixed Nash equilibrium (general)

Definition (Mixed Nash equilibrium (MNE))

A mixed strategy $\sigma_i : \mathcal{S}_i \to [0, 1]$ of player $i \in N$ is a probability distribution over pure strategies in $\mathcal{S}_i$, i.e., coming from

$$\Delta_i = \left\{ \tau : \tau(t) \geq 0 \ \ \forall t \in \mathcal{S}_i \quad \text{and} \quad \sum\nolimits_{t \in \mathcal{S}_i} \tau(t) = 1 \right\}.$$

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Here

- $\sigma : \times_j \mathcal{S}_j \to \mathbb{R}_{\geq 0}$ is given by $\sigma(t) = \prod_j \sigma_j(t_j)$, and
- $\sigma_{-i} : \times_{j \neq i} \mathcal{S}_j \to \mathbb{R}_{\geq 0}$ is given by $\sigma_{-i}(t_{-i}) = \prod_{j \neq i} \sigma_j(t_j)$.

Existence and computational complexity

Existence (“Nobel” Prize in Economics in 1994)

Theorem (Nash’s theorem, 1950)

Any finite game Γ *has a mixed Nash equilibrium.*

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Let $D \subseteq \mathbb{R}^m$ *be compact and convex, and let* $f : D \to D$ *be a continuous function. Then there exists an* $x^* \in D$ *such that* $f(x^*) = x^*$*.*

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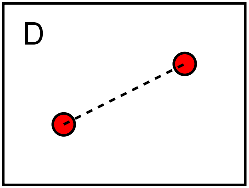
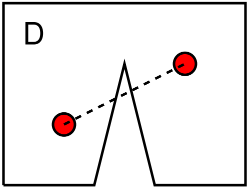
Convex means that line segments between points are included in D.



Convex



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Compact means **bounded** and **closed.**

- Satisfied by sets of mixed strategies Δ_i that we will be looking at.

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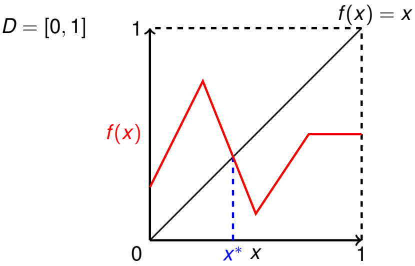
Brouwer's theorem says that f has a fixed point.



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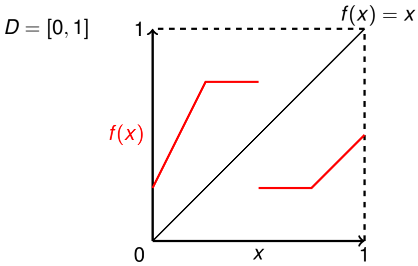
Brouwer's theorem fails if f is **not** continuous.



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Exercise: Show that $R_{z,s_z}(x,y)$ is a continuous function.

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and $y' \in \Delta_2$ by

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We use these functions to define mapping $f : \Delta_A \times \Delta_B \to \Delta_A \times \Delta_B$ by $f(x,y) = (x', y') = (x'_1, \dots, x'_m, y'_1, \dots, y'_n)$, where

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Exercise: Show that f is a continuous function.

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If (x^*, y^*) is MNE,

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If (x^*, y^*) is MNE, then $R_{z,s_z}(x,y) = 0 \;\; \forall z \in \{A, B\} \; \forall s_z \in \mathcal{S}_z$,

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If (x^*, y^*) is MNE, then $R_{z,s_z}(x,y) = 0 \ \forall z \in \{A, B\} \ \forall s_z \in \mathcal{S}_z$, and so $x' = x^*$ and $y' = y^*$.

$$R_{A,a_k}(x,y) = \max\{0, C_A(x,y) - C_A(e^k, y)\} \quad k = 1, \ldots, m$$
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Exercise: Show that f is a continuous function.

If (x^*, y^*) is MNE, then $R_{z,s_z}(x,y) = 0 \ \forall z \in \{A, B\} \ \forall s_z \in \mathcal{S}_z$, and so $x' = x^*$ and $y' = y^*$. In other words, (x^*, y^*) is fixed point of f.

Other direction remains: If (x^*, y^*) is fixed point of f, then it is MNE.

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Theorem (Nash's theorem, 1950)

Any finite game Γ *has a mixed Nash equilibrium.*

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(Context: Suppose $f(z) = z^2 + z - 2$, then $f(z) = z$ is solved by $z^* = \pm\sqrt{2}$.)

For $n \geq 3$,

- Rational ϵ-approximate MNE still exists for any $\epsilon > 0$.
- Algorithms are known to compute approx. equilibrium.
  - E.g., Scarf's algorithm (1967) for approximating fixed points.
  - Probably hard to compute in general (similar to upcoming discussion for $n = 2$).

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- See Chapter 20 [R2016] for this class, and more..

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There exists a constant $\epsilon > 0$ such that, assuming the "Exponential Time Hypothesis for PPAD", computing ϵ-approximate MNE in 2-player game requires time at least $n^{\log^{1-o(1)}(n)}$.

Two-player zero-sum games

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Algorithmic aspects of MNE:

- Can be modeled as optimal solution of linear program (LP).
  - Solvable in polynomial time.
  - *(Any LP can be written as zero-sum game as well.)*
- Certain player dynamics can "learn" it: Fictitious Play
  - Holds for more classes of games, but not in general.

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What can Alice guarantee to get from Bob?

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Consider a two-player zero-sum game given by matrix C. Then
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- Often referred to as the "Minimax theorem"

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- Exercise: Prove these corollaries

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Computing MNE using linear programming

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- We assume that the C is $m \times n$ matrix, i.e., m rows, n columns.

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\max & w & \\
\text{subject to} & w \leq \sum_{i=1}^{m} C_{ik} x_i & k = 1, \ldots, n \\
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& w \in \mathbb{R} &
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- For any fixed x, the number $\min_y x^T Cy$ is attained for some pure strategy e^k for $k = 1, \ldots, n$, where

$$
e^k_j = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} .
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  - In other words: $w \leq \min_y x^T Cy \quad \Leftrightarrow \quad w \leq x^T Ce_k \quad$ for $k = 1, \ldots, n$. *(Note that Ce^k is precisely the k-th column of the matrix C.)*

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Theorem

MNE can be computed in polynomial time in 2-player zero-sum game.

Two-player zero-sum games

Fictitious play

Simultaneous fictitious play (Brown, 1951)

Introduced as algorithm for approximating value of zero-sum game.

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- Analogous definition for Bob (with chosen column c_t in round t).

Example

Suppose the matrix C has $n = 6$ rows, and that Alice plays $(a_1, a_1, a_4, a_6, a_4, a_5, a_2, a_3, a_4)$ in first $t - 1 = 9$ rounds. Then

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Fictitious play algorithm

ALGORITHM 1: Fictitious play (with index tie-breaking rule)

Input : $m \times n$ matrix C; initial row r, column c; round total $T \in \mathbb{N}$.
Output: Empirical distributions $\bar{x}(T), \bar{y}(T)$.

$\bar{x}(1) = e_r$ and $\bar{y}(1) = e_c$.
for $t = 2, \ldots, T$ **do**
  Choose $r_t \in \text{argmax}\{(e^i)^T C\bar{y}(t) : i = 1, \ldots, m\}$
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  (Choose lowest indexed row/column in case of multiple best responses.)
  Update empirical distributions $(\bar{x}(t), \bar{y}(t))$ to $(\bar{x}(t+1), \bar{y}(t+1))$
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return $\bar{x}(T), \bar{y}(T)$

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- Observe that we specify a tie-breaking rule that decides which column/row to choose, in case there are multiple best responses.

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- Simple way to compute value and ϵ-MNE.
  - Avoiding the need to solve LPs.

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Empirical distributions $(\bar{x}(t), \bar{y}(t))$ "converge" to MNE as $t \to \infty$.

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