Pieter Kleer Winter 2020-2021

Topics in Algorithmic Game Theory and Economics, Exercise Sheet 1

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/game-theory

Total Points: 3 + (5 + 5 + 10 + 10) = 33 Due: Thursday, **Nov. 26, 23:59 (CET)**, 2020

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, using your own words. ***Always explain your answers.*** *Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of the total points on exercise sheets to be admitted to the exam. Send your solutions in* ***PDF*** *format directly to Golnoosh (**gshahkar@mpi-inf.mpg.de**). You get the first 3 points if you hand in typed solutions.*

Exercise 1 — 5 points

Give an example of a finite game $\Gamma = (N, (\mathcal{S}_i), (C_i))$ with the following two properties:

i) A unique pure Nash equilibrium.

ii) An (infinite) sequence of improving moves that does not converge to the unique pure Nash equilibrium in i).

Exercise 2 — 2+3 points

Let $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ be the symmetric network (atomic selfish routing) congestion game played on the graph G in Figure 1. In this case the (common) strategy set $\mathcal{S}_i$ of every player consists of the two s,t-paths in G.

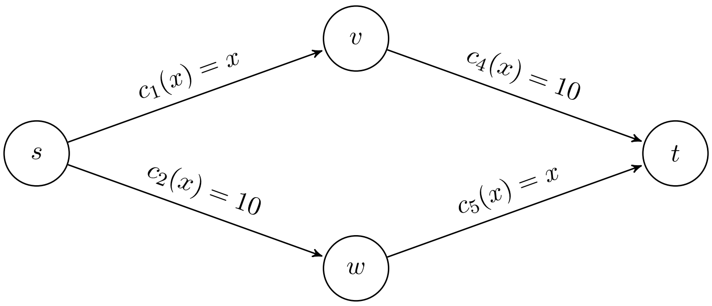

Figure 1: Graph G of Exercise 2.

We assume the game has $|N| = 10$ players. Let

$$C(s) = \sum_{i \in N} C_i(s),$$

be the total cost of strategy profile $s = (s_1, \dots, s_n) \in \times_i \mathcal{S}_i$ with $C_i(s) = \sum_{e \in s_i} c_e(x_e(s))$, and $x_e(s)$ the number of players using edge $e \in E$ in profile s.

a) Compute a pure Nash equilibrium of the above game.

b) Add a directed edge to the network (with non-negative cost function of your choice) such that there is a pure Nash equilibrium of the new game with higher total travel time than the equilibrium in part a). **(This is sometimes called a traffic paradox. Do you understand why?)**

Exercise 3 — 4+6 points

Consider the following finite ***anti-coordination*** game $(N, (\mathcal{S}_i), (U_i))$ played on the undirected graph $G = (N, E)$ with edge weights $w_e \geq 0$ for every $e \in E$. Each node $i \in N$ of G corresponds to a player who can choose either color red (r) or green (g), i.e., $\mathcal{S}_i = \{r, g\}$. Every player $i \in N$ has a parameter $\gamma_i > 0$ that determines the relative share that i obtains from the weights of incident edges to neighbours that choose a **different** color than i in a strategy profile s.

More precisely, given a strategy profile $s = (s_1, \dots, s_n) \in \{r, g\}^n$, the utility $U_i(s)$ of player i is defined as

$$U_i(s) = \sum_{e=\{i,j\} \in E:\ s_i \neq s_j} \frac{\gamma_i}{\gamma_i + \gamma_j} w_e.$$

A player's goal is to maximize her utility (or equivalently minimize her cost $C_i = -U_i$)

a) Show that this game is an exact potential game when $\gamma_i = \gamma_j$ for all $i, j \in N$. (Either work with utilities U_i or costs $C_i = -U_i$.)

b) Show that for general values of $\gamma_i > 0$, the above anti-coordination game always has at least one pure Nash equilibrium.

   (Hint: Try to generalize the potential function of part a). Be aware that the resulting potential function no longer satisfies $U_i(s_{-i}, s_i') - U_i(s) = \Phi(s_{-i}, s_i') - \Phi(s)$, but it does satisfy another property that is sufficient to apply the potential function method.)

Exercise 4 — 5+5 points

In a ***base matroid congestion game*** $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ the strategy set $\mathcal{S}_i$ of every player is the set of bases $\mathcal{B}_i$ of a matroid $\mathcal{M}_i = (E, \mathcal{I}_i)$ over the ground set E. *(Think, e.g., of spanning trees in a given undirected graph.)*

a) Let $s = (B_1, \dots, B_n)$, with $B_i \in \mathcal{B}_i$, be a strategy profile and suppose that $B_i' \in \mathcal{B}_i \setminus \{B_i\}$ is a best response for player i.

   Show that there exists a bijection $g : B_i' \setminus B_i \to B_i \setminus B_i'$ such that

$$c_a(x_a(s) + 1) \leq c_{g(a)}(x_{g(a)}(s))$$

   for every $a \in B_i' \setminus B_i$. *(Hint: Use a result from the background material on matroids.)*

b) Show, using part a), that any sequence of best response dynamics converges to a pure Nash equilibrium in at most $O(n^2 m^2)$ steps, where $n = |N|$ and $m = |E|$.