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Topics in Algorithmic Game Theory and Economics, Exercise Sheet 2

https://www.mpi-inf.mpg.de/departments/algorithms-complexity/teaching/winter20/game-theory

Total Points: 3 + 30 = 33 Due: Thursday, **Dec. 10, 23:59 (CET)**, 2020

You are allowed to collaborate on the exercise sheets, but you have to write down a solution on your own, using your own words. ***Always explain your answers****. Please indicate the names of your collaborators for each exercise you solve. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of the total points on exercise sheets to be admitted to the exam. Send your solutions in* ***PDF*** *format directly to Golnoosh (****gshahkar@mpi-inf.mpg.de****). You get the first 3 points if you hand in typed solutions.*

Exercise 1 — 5 points

Let $\mathcal{G}_n$ be the class of congestion games $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ with non-negative (meaning $c_e(x) \geq 0$ for all $x \in \mathbb{R}_{\geq 0}$) and non-decreasing (meaning $c_e(x) \leq c_e(y)$ whenever $x \leq y$) cost functions for $e \in E$, and precisely $n = |N|$ players. Show that $\text{PoS}(\mathcal{G}_n) = \sup_{\Gamma \in \mathcal{G}_n} \text{PoS}(\Gamma) = n$.

This means that you have to show that for any $\Gamma \in \mathcal{G}_n$ it holds that $\text{PoS}(\Gamma) \leq n$, and you have to show that for any $\epsilon > 0$, there exists a game $\Gamma \in \mathcal{G}_n$ for which $\text{PoS}(\Gamma) \geq n - \epsilon$.

Exercise 2 — 5 points

Let $\mathcal{G}^d$ be set of all congestion games $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ with cost functions polynomials of degree at most d, i.e.,

$$c_e(x) = a_{e,d}x^d + a_{e,d-1}x^{d-1} + \cdots + a_{e,1}x + a_{e,0}$$

with $a_{e,j} \geq 0$ for every $j = 0, \ldots, d$ and $e \in E$. Give an example of a game $\Gamma \in \mathcal{G}^d$ such that $\text{PoA}(\Gamma) \geq 2^d$. *(Hint: Modify the lower bound example for affine congestion games given during the lecture.)*

Exercise 3 — 4 + 4 points

Consider the class $\mathcal{G}$ of singleton congestion games $\Gamma = (N, E, (\mathcal{S}_i), (c_e))$ with affine cost functions. Here $\mathcal{S}_i \subseteq \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$ for every $i \in N$ and the cost functions are given by $c_e(x) = a_e x + b_e$ with $a_e, b_e \geq 0$ for every $e \in E$. Furthermore, let $\mathcal{G}_{\text{symm}} \subseteq \mathcal{G}$ be the subclass of instances with symmetric strategy sets, i.e., the games Γ for which $\mathcal{S}_i = \{\{e_1\}, \{e_2\}, \ldots, \{e_m\}\}$ for every $i \in N$.

a) Show that for any $\epsilon > 0$, there exists an example of a game $\Gamma \in \mathcal{G}_{\text{symm}}$ such that $\text{PoS}(\Gamma) \geq 4/3 - \epsilon$. (This implies that $\text{PoS}(\mathcal{G}_{\text{symm}}) \geq 4/3$.)

b) Give an example of a game $\Gamma \in \mathcal{G}$ such that $\text{PoA}(\Gamma) \geq 3/2$.

Some context: For symmetric singleton congestion games, it in fact holds that PoA($\mathcal{G}_{symm}$) = PoS($\mathcal{G}_{symm}$) = 4/3. For general (asymmetric) singleton congestion games, the lower bound of 3/2 can be improved to 5/2 (which matches the upper bound for general affine congestion games.)

Exercise 4 — 3 points

Consider the game

	b_1	b_2	b_3
a_1	$(5,1)$	$(3,10)$	$(7,7)$
a_2	$(2,4)$	$(2,-8)$	$(20,0)$
a_3	$(0,0)$	$(8,6)$	$(14,4)$

Is the pair (x, y) given by $x = (0, 1/3, 2/3)$ and $y = (3/4, 1/4, 0)$ an MNE? (Use any of the given definitions of MNE to prove your answer.)

Exercise 5 — 4 points

We showed that for a two-player zero-sum game given by matrix $C \in \mathbb{R}^{m\times n}$, the linear program

$$\begin{array}{lll} \max & w & \\ \text{subject to} & w \leq \sum_{i=1}^{m} C_{ij} x_i & j = 1, \ldots, n \\ & \sum_{i=1}^{m} x_i = 1 & \\ & x_i \geq 0 & i = 1, \ldots, m \\ & w \in \mathbb{R} & \end{array}$$

computes an optimal strategy for Alice (who tries to maximize her utility $x^T Cy$), and the optimum w is the value $v_A = v$ of the game. Give the linear program that computes an optimal strategy for Bob (who tries to minimize his cost $x^T Cy$).

Exercise 6 — 5 points

In this exercise we will be looking at the fictitious play algorithm. Consider the two-player zero-sum game given by the matrix

$$C = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

and assume that the initial row for Alice is a_1 and the initial column for Bob is b_1. We will use a different tie-breaking rule (than the index rule given during the lecture): If Alice is indifferent between choosing row a_1 and a_2 in step t (because their expected costs are equal given Bob's empirical distribution), then set

$$r_t = \begin{cases} a_1 & \text{if } r_{t-1} = a_2 \\ a_2 & \text{if } r_{t-1} = a_1 \end{cases}.$$

Similarly, set $c_t = \begin{cases} b_1 & \text{if } c_{t-1} = b_2 \\ b_2 & \text{if } c_{t-1} = b_1 \end{cases}$ in case of a tie between b_1 and b_2.

Write down the rows that Alice chooses in step $t = 1, \ldots, 20$, and the columns that Bob chooses in step $t = 1, \ldots, 20$. Where do the empirical probabilities (i.e., $\bar{x}_1(t)$, $\bar{x}_2(t)$, $\bar{y}_1(t)$ and $\bar{y}_2(t)$) converge to? (*Feel free to write a short implementation of the fictitious play algorithm.*)